

Addition theorems for spin spherical harmonics. II Results

Antonio O. Bouzas *

Departamento de Física Aplicada, CINVESTAV-IPN
Carretera Antigua a Progreso Km. 6, Apdo. Postal 73 “Cordemex”
Mérida 97310, Yucatán, México

January 14, 2013

Abstract

Based on the results of part I, we obtain the general form of the addition theorem for spin spherical harmonics and give explicit results in the cases involving one spin- s' and one spin- s spherical harmonics with $s', s = 1/2, 1, 3/2$, and $|s' - s| = 0, 1$. We obtain also a fully general addition theorem for one scalar and one tensor spherical harmonic of arbitrary rank. A variety of bilocal sums of ordinary and spin spherical harmonics are given in explicit form, including a general explicit expression for bilocal spherical harmonics.

1 Introduction

In this paper we study bilocal sums of spin spherical harmonics [1, 2] (see sect. 2 below for more details). Such expressions occur in the computation of physical observables in systems undergoing dynamical transitions induced by rotationally invariant spin-dependent interactions. Expanding the states involved in such transitions in a basis of total angular-momentum eigenfunctions leads to the sums considered here. In quantum systems, partial-wave expansions of transition amplitudes, or other matrix elements, are given by addition theorems for spin spherical harmonics. Another example is provided by multipolar expansions of the classical electromagnetic radiation field [3] in quadratic observables like energy, energy flow, or radiated power, which lead in some cases to bilocal sums of vector spherical harmonics.

Our main motivation for the present paper is the application of its results to the partial-wave analysis of scattering processes. In that context, partial waves satisfying two-body unitarity play an essential role in the theoretical approach (see [4] for a realistic application to hadronic reactions and an extensive reference list). Two-body intermediate states with different spins must be included in the analysis in order to satisfy unitarity, both as genuine asymptotic states or, in the case of resonances, as an approximation to higher n -body intermediate states. The addition theorems we consider in this paper are then needed for a systematic approach to partial-wave expansions of S -matrix elements between states involving particles with different spins. For that reason, we expect the results given below to be of interest also in other fields where quantum scattering is relevant. More generally, however, given the ubiquitous applications of spherical harmonics in all areas of physics as solutions to the Laplace equation, we hope the addition theorems discussed here to be of interest by themselves, independently of our specific motivations.

Our discussion of bilocal sums of spin spherical harmonics in the following sections is entirely based on results from the first part of this paper [5], hereafter referred to as I. We refer to equations and sections of the first part by prefixing their numbers with I. The matrix elements obtained in sect. I.3 are restated below as bilocal sums of ordinary spherical harmonics. In particular, they provide an explicit general expression for bilocal spherical harmonics [2]. The results of sect. I.3 also lead to a completely general addition theorem for, and to some classes of bilocal sums of, one scalar and one rank- n tensor spherical harmonic for any $n > 0$. Using the results of sects. I.2 and I.3, we obtain below a general addition theorem for two spin spherical harmonics with spins $s', s > 0$, and discuss it in completely explicit form in the cases $s', s \leq 3/2$, $|s' - s| = 0, 1$. Our approach can easily be applied to higher-spin spherical harmonics, although for $s > 2$ the size of the resulting algebraic expressions grows rapidly with s .

*E-mail: abouzas@mda.cinvestav.mx

The paper is organized as follows. In the next section we briefly review the definition and main properties of spin- s spherical harmonics, for any integer or half-integer s . In section 3 we study bilocal sums of ordinary spherical harmonics, including bilocal spherical harmonics. In section 4 we discuss bilocal sums involving one scalar and one tensor spherical harmonic. Addition theorems involving two spin spherical harmonics are considered in sect. 5. Finally, in section 6 we offer some final remarks. Throughout the paper we follow the notation and conventions of appendices I.A and I.B.

2 Spin spherical harmonics

Spin- s spherical harmonics $Y_{jj_z}^{\ell s}(\hat{r}) = \langle \hat{r}, \psi_s | \ell, s, j, j_z \rangle$ [1, 2] are eigenfunctions of \vec{L}^2 , \vec{S}^2 , \vec{J}^2 , J_z , appropriate for the description of a spin- s particle subject to spin-dependent central interactions. Expansion of the state $|\ell, s, j, j_z\rangle$ in the product basis $|\ell, \ell_z; s, s_z\rangle$ yields

$$Y_{jj_z}^{\ell s}(\hat{r}) = \sum_{\ell_z=-\ell}^{\ell} \sum_{s_z=-s}^s \langle \ell, \ell_z; s, s_z | j, j_z \rangle \langle \hat{r} | \ell, \ell_z \rangle \langle \psi_s | s, s_z \rangle, \quad (1)$$

where $|\hat{r}\rangle$ denotes a position eigenstate on the unit sphere, and $|\psi_s\rangle$ a basis spin state.

For integer spin $n > 0$ the spin wave functions are given by the standard rank- n irreducible tensors defined in sect. I.B.1, $\langle \psi_s | s, s_z \rangle = \hat{\varepsilon}^{i_1 \dots i_n}(s_z)$, $-s \leq s_z \leq s$, so that

$$(Y_{jj_z}^{\ell n}(\hat{r}))^{i_1 \dots i_n} = \sum_{s_z=-n}^n \sum_{\ell_z=-\ell}^{\ell} \langle \ell, \ell_z; n, s_z | j, j_z \rangle Y_{\ell \ell_z}(\hat{r}) \hat{\varepsilon}^{i_1 \dots i_n}(s_z). \quad (2)$$

For $n = 1$ this equality reduces to the definition of vector spherical harmonics familiar from textbooks [3, 6]. For $n > 1$, spin- n spherical harmonics are totally symmetric and traceless, as shown in appendix I.B.1. From that appendix we also obtain,

$$\begin{aligned} (Y_{jj_z}^{\ell n}(\hat{r}))^{i_1 \dots i_n *} &= (-1)^{j_z} (-1)^{\ell+n-j} \left(Y_{j(-j_z)}^{\ell n}(\hat{r}) \right)^{i_1 \dots i_n}, \\ \int d^2 p (Y_{jj_z}^{\ell n}(\hat{p}))^{i_1 \dots i_n *} (Y_{j'j'_z}^{\ell' n}(\hat{p}))^{i_1 \dots i_n} &= \delta_{\ell' \ell} \delta_{j'_z j_z} \delta_{j' j}, \\ \sum_{j, \ell=0}^{\infty} \sum_{j_z=-j}^j (Y_{jj_z}^{\ell n}(\hat{r}))^{i_1 \dots i_n} (Y_{jj_z}^{\ell n}(\hat{r}'))^{j_1 \dots j_n *} &= \delta(\hat{r} - \hat{r}') X^{i_1 \dots i_n; j_1 \dots j_n}, \end{aligned} \quad (3)$$

where in the last equality $X^{i_1 \dots i_n; j_1 \dots j_n}$ is the orthogonal projector onto the subspace of irreducible tensors of $\mathbb{C}^{3 \times \dots \times 3}$ defined in (I.B.9).

For half-integer spin, $s = n + 1/2$, $n \geq 0$, the spin wave functions are given by the standard spinors defined in sect. I.B.2, $\langle \psi_{n+1/2} | n + 1/2, s_z \rangle = \hat{\chi}_A^{i_1 \dots i_n}(s_z)$, which leads to spin- $(n + 1/2)$ spherical harmonics defined as

$$\left(Y_{jj_z}^{\ell(n+1/2)}(\hat{r}) \right)_A^{i_1 \dots i_n} = \sum_{s_z=-n-1/2}^{n+1/2} \sum_{\ell_z=-\ell}^{\ell} \langle \ell, \ell_z; n + 1/2, s_z | j, j_z \rangle Y_{\ell \ell_z}(\hat{r}) \hat{\chi}_A^{i_1 \dots i_n}(s_z). \quad (4)$$

For $n = 0$ we have the definition of spin-1/2 spherical harmonics, possessing two independent complex components. For $n > 0$, from appendix (I.B.2) we have that spin- $(n + 1/2)$ spherical harmonics are totally symmetric and traceless in their tensor indices and satisfy,

$$\sigma_{AB}^{i_k} \left(Y_{jj_z}^{\ell(n+1/2)}(\hat{r}) \right)_B^{i_1 \dots i_k \dots i_n} = 0, \quad 1 \leq k \leq n. \quad (5)$$

From the properties of spinors in appendix (I.B.2) we also obtain,

$$\begin{aligned} i\sigma_{AB}^2 \left(Y_{jj_z}^{\ell(n+1/2)}(\hat{r}) \right)_B^{i_1 \dots i_n} &= (-1)^{\ell+n+1/2-j} (-1)^{1/2+j_z} \left(Y_{j(-j_z)}^{\ell(n+1/2)}(\hat{r}) \right)_B^{i_1 \dots i_n *}, \\ \int d^2 p \left(Y_{jj_z}^{\ell(n+1/2)}(\hat{p}) \right)_A^{i_1 \dots i_n *} \left(Y_{j'j'_z}^{\ell'(n+1/2)}(\hat{p}) \right)_A^{i_1 \dots i_n} &= \delta_{\ell' \ell} \delta_{j'_z j_z} \delta_{j' j}, \\ \sum_{j, \ell=0}^{\infty} \sum_{j_z=-j}^j \left(Y_{jj_z}^{\ell(n+1/2)}(\hat{r}) \right)_A^{i_1 \dots i_n} \left(Y_{jj_z}^{\ell(n+1/2)}(\hat{r}') \right)_B^{j_1 \dots j_n *} &= \delta(\hat{r} - \hat{r}') X_{AB}^{i_1 \dots i_n; j_1 \dots j_n}, \end{aligned} \quad (6)$$

where $X_{AB}^{i_1 \dots i_n; j_1 \dots j_n}$ is the orthogonal projector onto the spin- $(n + 1/2)$ subspace of $\mathbb{C}^{3 \times \dots \times 3 \times 2}$ defined in (I.B.23).

3 Bilocal sums of two scalar spherical harmonics

The matrix elements of the angular-momentum projector operators with tensor products of \vec{L} and \hat{r} obtained in section I.3 constitute addition theorems, or weighted sums, for two scalar spherical harmonics. In this section we summarize the main results.

From the definition (I.52) of \mathcal{P}_ℓ we have,

$$\begin{aligned} \sum_{\ell'_z = -\ell_n}^{\ell_n} \sum_{\ell_z = -\ell}^{\ell} Y_{\ell_n \ell'_z}(\hat{p}') Y_{\ell \ell_z}(\hat{p})^* \langle \ell_n, \ell'_z | \hat{r}^{\{i_1 \dots i_n\}} L^{j_1} \dots L^{j_t} \rangle_0 | \ell, \ell_z \rangle = \\ = \langle \hat{p}' | \mathcal{P}_{\ell_n} \hat{r}^{\{i_1 \dots i_n\}} L^{j_1} \dots L^{j_t} \rangle_0 \mathcal{P}_\ell | \hat{p} \rangle. \end{aligned} \quad (7)$$

The matrix element on the r.h.s. of (7) is given explicitly for any $\ell \geq 0$ by (I.65) and (I.66) for $n > 0, t > 0$, by (I.56) and (I.57) for $n = 0, t > 0$, and by (I.60) and (I.61) for $n > 0, t = 0$. Notice that, in the last case, the matrix element in (7) involves an additional symmetrization with respect to (I.60). The equalities (I.65), (I.56) and (I.60) written in the form (7) will be needed below in the derivation of addition theorems for spin spherical harmonics. By multiplying (7) by a standard tensor of rank $n + s$, as defined in appendix (I.B), and applying the Wigner-Eckart theorem [1, 2, 6–10] to the matrix element on the l.h.s., we obtain the equivalent form

$$\begin{aligned} \sum_{\ell_z = -\ell}^{\ell} Y_{\ell_n(\ell_z+m)}(\hat{p}') Y_{\ell \ell_z}(\hat{p})^* \langle \ell, \ell_z; n+t, m | \ell_n, \ell_z+m \rangle = \\ = \frac{1}{(n+t)!} \hat{\varepsilon}^{h_1 \dots h_n k_1 \dots k_t}(m) \frac{\langle \hat{p}' | \mathcal{P}_{\ell_n} \hat{r}^{\{h_1 \dots h_n\}} L^{k_1} \dots L^{k_t} \rangle_0 \mathcal{P}_\ell | \hat{p} \rangle}{\langle \ell_n | \hat{\varepsilon}^{a_1 \dots a_n b_1 \dots b_t} \hat{r}^{a_1} \dots \hat{r}^{a_n} L^{b_1} \dots L^{b_t} | \ell \rangle}, \end{aligned} \quad (8)$$

where the reduced matrix element in the denominator on the r.h.s. is given by (I.C.8) for $n > 0, t > 0$, by (I.C.3) for $n = 0, t > 0$, and by (I.C.6) for $n > 0, t = 0$. Equation (8) will be recast below in sect. 4 in the form of an addition theorem for one scalar and one tensor spherical harmonic.

Equations (7) and (8) contain a great deal of information. First, together with the matrix elements given explicitly in I, (8) provides an explicit representation for bilocal spherical harmonics [2], the wave functions associated to coupled orbital angular momentum states $|\ell, \ell', j, j_z\rangle$

$$\begin{aligned} \langle \hat{p}, \hat{p}' | \ell, \ell', n, m \rangle = \sum_{\ell'_z = -\ell'}^{\ell'} \sum_{\ell_z = -\ell}^{\ell} Y_{\ell' \ell'_z}(\hat{p}') Y_{\ell \ell_z}(\hat{p}) \langle \ell, \ell_z; \ell', \ell'_z | n, m \rangle \\ = \frac{(-1)^{n-\ell'}}{n!} \sqrt{\frac{2n+1}{2\ell'+1}} \hat{\varepsilon}^{h_1 \dots h_n}(m) \frac{\langle \hat{p}' | \mathcal{P}_{\ell'} \hat{r}^{\{h_1 \dots h_n\}} L^{h_{|\Delta \ell|}} L^{h_{|\Delta \ell|+1}} \dots L^{h_n} \rangle_0 \mathcal{P}_\ell | \hat{p} \rangle}{\langle \ell' | \hat{\varepsilon}^{a_1 \dots a_n} \hat{r}^{a_1} \dots \hat{r}^{a_n} L^{a_{|\Delta \ell|}} L^{a_{|\Delta \ell|+1}} \dots L^{a_n} | \ell \rangle}. \end{aligned}$$

Second, seen as a generic sum of products of two spherical harmonics weighted by a CG coefficient, (8) is best encoded in terms of tensor spherical harmonics, as done in the following section. Finally, by considering the explicit expression of the CG on its l.h.s. as a function of ℓ_z , we can interpret (8) as a sum of bilocal products of two spherical harmonics weighted by specific functions. We discuss several such bilocal sums in the remainder of this section.

If we set $n = 0$, (7) involves only matrix elements of \vec{L} . We will discuss here two classes of bilocal sums obtained from (7) in that case. First, we consider moments of ℓ_z . We have

$$\begin{aligned} \langle \ell, \ell_z | L^{\{3 \dots 3\}} | \ell, \ell_z \rangle = t! \hat{\varepsilon}^{3 \dots 3}(0)^* \langle \ell | \hat{\varepsilon}^{j_1 \dots j_t} L^{j_1} \dots L^{j_t} | \ell \rangle \langle \ell, \ell_z; t, 0 | \ell, \ell_z \rangle \\ = \frac{(t!)^2}{2^t (2t-1)!!} \frac{1}{\sqrt{2\ell+1}} \sqrt{\frac{(2\ell+t+1)!}{(2\ell-t)!}} \langle \ell, \ell_z; t, 0 | \ell, \ell_z \rangle, \end{aligned} \quad (9)$$

where the second equality follows from (I.B.7) applied to $\hat{\varepsilon}^{3 \dots 3}(m)$. The CG coefficient on the r.h.s. of (9) is a polynomial of degree t in ℓ_z , so substituting (9) on the l.h.s. of (7) we obtain the desired moments. For $t \leq 3$

the irreducible tensors appearing in the r.h.s. of (7) (see (I.56a)) are easily expanded to get, with the notation of (I.56b)

$$\sum_{\ell_z=-\ell}^{\ell} \ell_z Y_{\ell\ell_z}(\hat{p}') Y_{\ell\ell_z}(\hat{p})^* = i \frac{2\ell+1}{4\pi} v^3 P'_\ell(x), \quad (10a)$$

$$\sum_{\ell_z=-\ell}^{\ell} \ell_z^2 Y_{\ell\ell_z}(\hat{p}') Y_{\ell\ell_z}(\hat{p})^* = -\frac{2\ell+1}{4\pi} \left((v^3)^2 P''_\ell(x) + (\hat{p}^3 \hat{p}'^3 - x) P'_\ell(x) \right), \quad (10b)$$

$$\sum_{\ell_z=-\ell}^{\ell} \ell_z^3 Y_{\ell\ell_z}(\hat{p}') Y_{\ell\ell_z}(\hat{p})^* = -i \frac{2\ell+1}{4\pi} v^3 \left((v^3)^2 P'''_\ell(x) + 3(\hat{p}^3 \hat{p}'^3 - x) P''_\ell(x) - P'_\ell(x) \right), \quad (10c)$$

where the Legendre equation was used to simplify the right-hand sides. For $t > 3$ the relations among tensors and their irreducible components become too lengthy, so we do not expand them

$$\begin{aligned} \sum_{\ell_z=-\ell}^{\ell} \ell_z^4 Y_{\ell\ell_z}(\hat{p}') Y_{\ell\ell_z}(\hat{p})^* &= \frac{2\ell+1}{4\pi} \frac{1}{4!} \left(v^{\{3v^3v^3v^3\}_0} P_\ell^{(4)}(x) + 6\hat{p}^{\{3\hat{p}'^3v^3v^3\}_0} P_\ell^{(3)}(x) \right. \\ &\quad + 3\hat{p}^{\{3\hat{p}'^3\hat{p}'^3\hat{p}'^3\}_0} P_\ell^{(2)}(x) - \frac{12}{7}(6\ell(\ell+1)-5) \left(v^{\{3v^3\}_0} P_\ell^{(2)}(x) + \hat{p}^{\{3\hat{p}'^3\}_0} P_\ell^{(1)}(x) \right) \\ &\quad \left. + \frac{8}{5}\ell(\ell+1)(3\ell(\ell+1)-1)P_\ell(x) \right), \end{aligned} \quad (10d)$$

$$\begin{aligned} \sum_{\ell_z=-\ell}^{\ell} \ell_z^5 Y_{\ell\ell_z}(\hat{p}') Y_{\ell\ell_z}(\hat{p})^* &= i \frac{2\ell+1}{4\pi} \frac{1}{5!} \left(v^{\{3v^3v^3v^3v^3\}_0} P_\ell^{(5)}(x) + 10\hat{p}^{\{3\hat{p}'^3v^3v^3v^3\}_0} P_\ell^{(4)}(x) \right. \\ &\quad + 15\hat{p}^{\{3\hat{p}'^3\hat{p}'^3\hat{p}'^3v^3\}_0} P_\ell^{(2)}(x) - \frac{100}{9}(2\ell(\ell+1)-3) \left(v^{\{3v^3v^3\}_0} P_\ell^{(3)}(x) + 3\hat{p}^{\{3\hat{p}'^3v^3\}_0} P_\ell^{(2)}(x) \right) \\ &\quad \left. + \frac{120}{7}(3\ell^2(\ell+1)^2 - 3\ell(\ell+1) + 1)v^3 P_\ell^{(1)}(x) \right). \end{aligned} \quad (10e)$$

In the last two equalities the tensors on the r.h.s. may be numerically evaluated as, e.g., $v^{\{3v^3v^3v^3\}_0} = 4!\hat{\varepsilon}^{3333}(0)^*\hat{\varepsilon}^{i_1i_2i_3i_4}(0)v^{i_1}v^{i_2}v^{i_3}v^{i_4}$, as we actually have done for numerical verifications of (10d), (10e).

A second class of bilocal sums is obtained from (7) with $n = 0$ by multiplying both sides by $\hat{\varepsilon}^{i_1\dots i_t}(\pm t)$ and using (I.B.11) and the known matrix elements of $L^\pm = L^1 \pm iL^2$. Thus,

$$\begin{aligned} \sum_{\ell_z=-\ell}^{\ell} Y_{\ell(\ell_z\pm t)}(\hat{p}') Y_{\ell\ell_z}(\hat{p})^* \sqrt{\frac{(\ell \mp \ell_z)!}{(\ell \mp \ell_z - t)!}} \sqrt{\frac{(\ell \pm \ell_z + t)!}{(\ell \pm \ell_z)!}} &= \\ \sum_{\ell_z=-\ell}^{\ell} Y_{\ell(\ell_z\pm t)}(\hat{p}') Y_{\ell\ell_z}(\hat{p})^* \langle \ell, \ell_z; t, \pm t | \ell, \ell_z \pm t \rangle (\pm 1)^t 2^{\frac{t}{2}} \langle \ell | \hat{\varepsilon}^{i_1\dots i_t} L^{i_1} \dots L^{i_t} | \ell \rangle &= \\ = i^t \frac{2\ell+1}{4\pi} \sum_{q=0}^{[t/2]} C_{t,q}(\hat{p}^\pm)^q (\hat{p}'^\pm)^q (v^\pm)^{t-2q} P_\ell^{(t-q)}(x), \end{aligned} \quad (11)$$

with $C_{t,q}$ as defined in (I.56), $y^\pm = y^1 \pm iy^2 = \mp\sqrt{2}\hat{\varepsilon}(\pm 1) \cdot \vec{y}$ for any \vec{y} , and with $Y_{\ell m} \equiv 0$ if $|m| > \ell$. It is understood in (11) that we must choose either all upper signs or all lower ones.

Next, we consider (7) with $t = 0$, so only matrix elements of \hat{r} are involved. In that situation (7) generalizes to the bilocal case some results known in the local case $\hat{p}' = \hat{p}$ (see [2] and refs. therein). We give here their explicit form for $n = 1, 2$. For $n = 1$, $t = 0$ we have,

$$\sum_{\ell_z=-\ell}^{\ell} Y_{\ell_1\ell_z}(\hat{p}') Y_{\ell\ell_z}(\hat{p})^* \sqrt{(\langle \ell \rangle + 1/2)^2 - \ell_z^2} = \Delta\ell \frac{1}{4\pi} \sqrt{2\ell+1} \sqrt{2\ell_1+1} (\hat{p}'^3 P'_{\ell_1}(x) - \hat{p}^3 P'_\ell(x)), \quad (12a)$$

$$\begin{aligned} \sum_{\ell_z=-\ell}^{\ell} Y_{\ell_1(\ell_z+m)}(\hat{p}') Y_{\ell\ell_z}(\hat{p})^* \sqrt{\ell_1 + \Delta\ell m \ell_z} \sqrt{\ell_1 + \Delta\ell m \ell_z + 1} &= \\ = -\frac{m}{4\pi} \sqrt{2\ell+1} \sqrt{2\ell_1+1} (\hat{p}'^m P'_{\ell_1}(x) - \hat{p}^m P'_\ell(x)), \end{aligned} \quad (12b)$$

where $m = \pm 1$. For $n = 2, t = 0$ we can write (7) as

$$\begin{aligned} \sum_{\ell_z = -\ell+1}^{\ell-1} Y_{(\ell+1)\ell_z}(\hat{p}') Y_{(\ell-1)\ell_z}(\hat{p})^* \sqrt{(\ell^2 - \ell_z^2)((\ell+1)^2 - \ell_z^2)} = \\ = \frac{1}{4\pi} \sqrt{(2\ell-1)(2\ell+3)} ((\hat{p}'^3)^2 P_{\ell+1}''(x) - 2\hat{p}'^3 \hat{p}'^3 P_{\ell}''(x) + (\hat{p}'^3)^2 P_{\ell-1}''(x) - P_{\ell}'(x)), \end{aligned} \quad (13a)$$

$$\begin{aligned} \sum_{\ell_z = -\ell+1}^{\ell-1} Y_{(\ell+1)(\ell_z+m)}(\hat{p}') Y_{(\ell-1)\ell_z}(\hat{p})^* \sqrt{(\ell^2 - \ell_z^2)} \sqrt{(\ell+m\ell_z+1)(\ell+m\ell_z+2)} = \\ = -\frac{m}{4\pi} \sqrt{(2\ell-1)(2\ell+3)} (\hat{p}'^m \hat{p}'^3 P_{\ell+1}''(x) - (\hat{p}'^m \hat{p}'^3 + \hat{p}'^3 \hat{p}'^m) P_{\ell}''(x) + \hat{p}'^m \hat{p}'^3 P_{\ell-1}''(x)), \end{aligned} \quad (13b)$$

$$\begin{aligned} \sum_{\ell_z = -\ell+1}^{\ell-1} Y_{(\ell+1)(\ell_z+2m)}(\hat{p}') Y_{(\ell-1)\ell_z}(\hat{p})^* \sqrt{\frac{(\ell+m\ell_z+3)!}{(\ell+m\ell_z-1)!}} = \\ = \frac{1}{4\pi} \sqrt{(2\ell-1)(2\ell+3)} ((\hat{p}'^m)^2 P_{\ell+1}''(x) - 2\hat{p}'^m \hat{p}'^m P_{\ell}''(x) + (\hat{p}'^m)^2 P_{\ell-1}''(x)), \end{aligned} \quad (13c)$$

where on the second and third equalities $m = \pm 1$, and $\hat{p}'^m = \hat{p}^1 + im\hat{p}^2$. For $n > 2$, similar but more involved results are obtained from (7). We will not discuss them further here for reasons of space.

Finally, we discuss some explicit forms of (7) for low values of $n, t > 0$, which extend (12) and (13) by insertion of powers of ℓ_z in the l.h.s. For $n = 1 = t$ with $m = 0, \pm 1$ we find

$$\sum_{\ell_z = -\ell}^{\ell} Y_{\ell_1 \ell_z}(\hat{p}') Y_{\ell \ell_z}(\hat{p})^* \ell_z \sqrt{(\langle \ell \rangle + 1/2)^2 - \ell_z^2} = i \frac{\Delta \ell}{4\pi} \sqrt{(2\ell+1)(2\ell_1+1)} v^3 (\hat{p}'^3 P_{\ell_1}''(x) - \hat{p}'^3 P_{\ell}''(x)), \quad (14a)$$

$$\begin{aligned} \sum_{\ell_z = -\ell}^{\ell} Y_{\ell_1(\ell_z+m)}(\hat{p}') Y_{\ell \ell_z}(\hat{p})^* \ell_z \sqrt{\frac{(\ell_1 + \Delta \ell m \ell_z + 1)!}{(\ell_1 + \Delta \ell m \ell_z - 1)!}} = -\frac{1}{8\pi} \sqrt{(2\ell+1)(2\ell_1+1)} \left[im(v^m \hat{p}'^3 + \right. \\ \left. + v^3 \hat{p}'^m) P_{\ell_1}''(x) - im(v^m \hat{p}'^3 + v^3 \hat{p}'^m) P_{\ell}''(x) + \Delta \ell (\ell + (1 - \Delta \ell)/2) (\hat{p}'^m P_{\ell_1}'(x) - \hat{p}'^m P_{\ell}'(x)) \right], \end{aligned} \quad (14b)$$

to be compared with (12). Similarly, for $n = 1, t = 2$, with $m = 0$ we get

$$\begin{aligned} \sum_{\ell_z = -\ell}^{\ell} Y_{\ell_1 \ell_z}(\hat{p}') Y_{\ell \ell_z}(\hat{p})^* \ell_z^2 \sqrt{(\langle \ell \rangle + 1/2)^2 - \ell_z^2} = -\frac{\Delta \ell}{20\pi} \sqrt{(2\ell+1)(2\ell_1+1)} \left[(-\hat{p}'^3(1-x^2) + 5\hat{p}'^3(v^3)^2) \right. \\ \times P_{\ell_1}'''(x) - (-\hat{p}'^3(1-x^2) + 5\hat{p}'^3(v^3)^2) P_{\ell}'''(x) + (-\hat{p}'^3 - 2x\hat{p}'^3 + 5\hat{p}'^3(\hat{p}'^3)^2) P_{\ell_1}''(x) \\ \left. - (-\hat{p}'^3 - 2x\hat{p}'^3 + 5\hat{p}'^3(\hat{p}'^3)^2) P_{\ell}''(x) - (\langle \ell \rangle - 1/2)(\langle \ell \rangle + 3/2) (\hat{p}'^3 P_{\ell_1}'(x) - \hat{p}'^3 P_{\ell}'(x)) \right], \end{aligned} \quad (15)$$

to be compared with (12a) and (14a). The analogous results for $m = \pm 1, \pm 2, \pm 3$ are somewhat lengthy, so we omit them for brevity. Finally, with $n = 2, t = 1$ and $m = 0$ we obtain

$$\begin{aligned} \sum_{\ell_z = -\ell+1}^{\ell-1} Y_{(\ell+1)\ell_z}(\hat{p}') Y_{(\ell-1)\ell_z}(\hat{p})^* \ell_z \sqrt{(\ell^2 - \ell_z^2)((\ell+1)^2 - \ell_z^2)} = \frac{1}{4\pi} \sqrt{(2\ell-1)(2\ell_1+3)} iv^3 \\ \times [(\hat{p}'^3)^2 P_{\ell+1}'''(x) - 2\hat{p}'^3 \hat{p}'^3 P_{\ell}'''(x) + (\hat{p}'^3)^2 P_{\ell-1}'''(x) - P_{\ell}''(x)], \end{aligned} \quad (16)$$

which is related to (13a). As in the previous case we omit the explicit results for $m \neq 0$, as well as those for higher n and t , for brevity. Those cases can be obtained from the general form (8), either algebraically or numerically. Analogous, but lengthier, results can be obtained by the same techniques for higher values of n and t .

Before leaving the subject of bilocal sums of spherical harmonics, we briefly discuss the restriction of the previous results to the local case $\hat{p}' = \hat{p}$. From (8) with (I.65), (I.B.17), (I.C.8) and (I.C.4) we get,

$$\begin{aligned} \sum_{\ell_z = -\ell}^{\ell} Y_{\ell_n(\ell_z+m)}(\hat{p}) Y_{\ell \ell_z}(\hat{p})^* \langle \ell, \ell_z; n+t, m | \ell_n, \ell_z+m \rangle = \\ = \sqrt{\frac{(2\ell+1)(2\ell_n+1)}{2(n+t)+1}} \langle \ell, 0; n+t, 0 | \ell_n, 0 \rangle \frac{1}{\sqrt{4\pi}} Y_{(n+t)m}(\hat{p}), \end{aligned} \quad (17)$$

valid for $n, t \geq 0$ and $|m| \leq n + t$. Notice that the r.h.s. vanishes for t odd, because the CG coefficient does. This is due to the fact that, as seen from (I.65), when $\hat{p}' = \hat{p}$ and t is odd all terms in the r.h.s. of (7), (8) contain a positive tensor power of $\vec{v} = 0$. Equation (17) is, of course, the inverse of the well-known Clebsch-Gordan series for spherical harmonics [2]. For particular values of its parameters it reduces to sums of spherical harmonics with certain weight functions. We list several of those particular cases in appendix A.

4 Addition theorems for one scalar and one tensor spherical harmonic

The results from the previous section for bilocal sums of spherical harmonics lead to a completely general addition theorem involving one scalar and one tensor spherical harmonic. Multiplying both sides of (8) by $\hat{\varepsilon}^{i_1 \dots i_n j_1 \dots j_t}(m)^*$, and summing over m , we get

$$\begin{aligned} \sum_{\ell'_z = -\ell_n}^{\ell_n} Y_{\ell_n \ell'_z}(\hat{p}') \left(Y_{\ell_n \ell'_z}^{\ell(n+t)}(\hat{p}) \right)^{i_1 \dots i_{n+t}}{}^* &= (-1)^n \sqrt{\frac{2\ell_n + 1}{2\ell + 1}} \sum_{\ell_z = -\ell}^{\ell} \left(Y_{\ell \ell_z}^{\ell(n+t)}(\hat{p}') \right)^{i_1 \dots i_{n+t}} Y_{\ell \ell_z}(\hat{p})^* \\ &= \frac{1}{(n+t)!} \frac{\langle \hat{p}' | \mathcal{P}_{\ell_n} \hat{r}^{\{i_1} \dots \hat{r}^{i_n} L^{i_{n+1}} \dots L^{i_{n+t}}\}0} \mathcal{P}_{\ell} | \hat{p} \rangle}{\langle \ell_n || \hat{\varepsilon}^{k_1 \dots k_{n+t}} \hat{r}^{k_1} \dots \hat{r}^{k_n} L^{k_{n+1}} \dots L^{k_{n+t}} || \ell \rangle}, \end{aligned} \quad (18)$$

with n, t nonnegative integers. The matrix elements on the r.h.s. of (18) are given explicitly by (I.65) and (I.C.8) for $n > 0, t > 0$, by (I.56) and (I.C.3) for $n = 0, t > 0$, and by (I.60) and (I.C.6) for $n > 0, t = 0$. Notice that, in the last case, the matrix element in (18) involves an additional symmetrization with respect to (I.60). We give here the expanded forms of (18) for spins $n + t = 1, 2, 3$ (see [2] for some related results).

For vector spherical harmonics (18) yields,

$$\sum_{\ell_z = -\ell}^{\ell} \left(Y_{\ell \ell_z}^{\ell 1}(\hat{p}') \right)^i Y_{\ell \ell_z}(\hat{p})^* = \frac{i}{4\pi} \frac{2\ell + 1}{\sqrt{\ell(\ell + 1)}} v^i P_{\ell}'(x), \quad (19a)$$

$$\sum_{\ell_z = -\ell}^{\ell} \left(Y_{\ell \ell_z}^{\ell 1}(\hat{p}') \right)^i Y_{\ell \ell_z}(\hat{p})^* = -\frac{\sqrt{2}}{4\pi} \sqrt{\frac{2\ell + 1}{\ell_1 + \ell + 1}} \left(\hat{p}'^i P_{\ell_1}'(x) - \hat{p}^i P_{\ell}'(x) \right). \quad (19b)$$

For rank-2 tensor spherical harmonics, by setting $n + t = 2$ in (18) we get,

$$\sum_{\ell_z = -\ell}^{\ell} \left(Y_{\ell \ell_z}^{\ell 2}(\hat{p}') \right)^{i_1 i_2} Y_{\ell \ell_z}(\hat{p})^* = -\frac{\sqrt{3/2} (2\ell + 1) / \sqrt{\ell(\ell + 1)}}{4\pi \sqrt{(2\ell - 1)(2\ell + 3)}} \left(v^{\{i_1} v^{i_2\}0} P_{\ell}''(x) + \hat{p}'^{\{i_1} \hat{p}^{i_2\}0} P_{\ell}'(x) \right), \quad (20a)$$

$$\sum_{\ell_z = -\ell}^{\ell} \left(Y_{\ell \ell_z}^{\ell 2}(\hat{p}') \right)^{i_1 i_2} Y_{\ell \ell_z}(\hat{p})^* = -\frac{i}{2\pi} \sqrt{2\ell + 1} \sqrt{\frac{(\ell_1 + \ell - 3)!!}{(\ell_1 + \ell + 3)!!}} \left(\hat{p}'^{\{i_1} v^{i_2\}} P_{\ell_1}''(x) - \hat{p}^{\{i_1} v^{i_2\}} P_{\ell}''(x) \right), \quad (20b)$$

$$\begin{aligned} \sum_{\ell_z = -\ell}^{\ell} \left(Y_{\ell \ell_z}^{\ell 2}(\hat{p}') \right)^{i_1 i_2} Y_{\ell \ell_z}(\hat{p})^* &= \frac{1}{4\pi} \frac{1}{\sqrt{2\ell_2 + 1}} \sqrt{\frac{(\ell_2 + \ell - 1)(\ell_2 + \ell + 3)}{(\ell_2 + \ell)(\ell_2 + \ell + 1)(\ell_2 + \ell + 2)}} \\ &\times \left(\hat{p}'^{\{i_1} \hat{p}^{i_2\}0} P_{\ell_2}''(x) - 2\hat{p}'^{\{i_1} \hat{p}^{i_2\}0} P_{\ell_1}''(x) + \hat{p}^{\{i_1} \hat{p}^{i_2\}0} P_{\ell}''(x) \right). \end{aligned} \quad (20c)$$

For rank-3 tensor spherical harmonics, setting $n + t = 3$ in (18) yields,

$$\begin{aligned} \sum_{\ell_z = -\ell}^{\ell} \left(Y_{\ell \ell_z}^{\ell 3}(\hat{p}') \right)^{i_1 i_2 i_3} Y_{\ell \ell_z}(\hat{p})^* &= -\frac{i}{2\pi} \frac{\sqrt{10}}{3} \sqrt{\frac{(2\ell - 3)!}{(2\ell + 4)!}} (2\ell + 1)^{3/2} \\ &\times \left(v^{\{i_1} v^{i_2} v^{i_3\}0} P_{\ell}^{(3)}(x) + 3\hat{p}'^{\{i_1} \hat{p}^{i_2} v^{i_3\}0} P_{\ell}^{(2)}(x) \right), \end{aligned} \quad (21a)$$

$$\begin{aligned} \sum_{\ell_z = -\ell}^{\ell} \left(Y_{\ell \ell_z}^{\ell 3}(\hat{p}') \right)^{i_1 i_2 i_3} Y_{\ell \ell_z}(\hat{p})^* &= \frac{1}{4\pi} \sqrt{\frac{10}{3}} \sqrt{2\ell + 1} \sqrt{\frac{(\ell_1 + \ell - 3)!}{(\ell_1 + \ell + 4)!}} \sqrt{(\ell_1 + \ell)(\ell_1 + \ell + 2)} \\ &\times \left(\hat{p}'^{\{i_1} v^{i_2} v^{i_3\}0} P_{\ell_1}^{(3)}(x) - \hat{p}^{\{i_1} v^{i_2} v^{i_3\}0} P_{\ell}^{(3)}(x) + \hat{p}'^{\{i_1} \hat{p}^{i_2} \hat{p}^{i_3\}0} P_{\ell_1}^{(2)}(x) - \hat{p}'^{\{i_1} \hat{p}^{i_2} \hat{p}^{i_3\}0} P_{\ell}^{(2)}(x) \right), \end{aligned} \quad (21b)$$

$$\sum_{\ell_z=-\ell}^{\ell} \left(Y_{\ell\ell_z}^{\ell_2 3}(\hat{p}') \right)^{i_1 i_2 i_3} Y_{\ell\ell_z}(\hat{p})^* = \frac{i}{8\pi} \frac{1}{\sqrt{3}} \frac{1}{(2\ell_1+1)\sqrt{2\ell_2+1}} \sqrt{\frac{(2\ell_1+3)!!}{(2\ell_1-3)!!}} \sqrt{\frac{(\ell_1-2)!}{(\ell_1+2)!}} \quad (21c)$$

$$\times \left(\hat{p}'^{\{i_1} \hat{p}'^{i_2} v^{i_3\} P_{\ell_2}^{(3)}(x) - 2\hat{p}'^{\{i_1} \hat{p}'^{i_2} v^{i_3\} P_{\ell_1}^{(3)}(x) + \hat{p}'^{\{i_1} \hat{p}'^{i_2} v^{i_3\} P_{\ell}^{(3)}(x) \right),$$

$$\sum_{\ell_z=-\ell}^{\ell} \left(Y_{\ell\ell_z}^{\ell_3 3}(\hat{p}') \right)^{i_1 i_2 i_3} Y_{\ell\ell_z}(\hat{p})^* = -\frac{1}{3\pi} \frac{\sqrt{2}\sqrt{\ell}+2}{(2\ell_1+1)(2\ell_2+1)\sqrt{2\ell_3+1}} \sqrt{\frac{(\ell_3+\ell-3)!!}{(\ell_3+\ell+3)!!}} \sqrt{\frac{(\ell+1)!}{(\ell-2)!}} \quad (21d)$$

$$\times \left(\hat{p}'^{\{i_1} \hat{p}'^{i_2} \hat{p}'^{i_3\} P_{\ell_3}^{(3)}(x) - 3\hat{p}'^{\{i_1} \hat{p}'^{i_2} \hat{p}'^{i_3\} P_{\ell_2}^{(3)}(x) + 3\hat{p}'^{\{i_1} \hat{p}'^{i_2} \hat{p}'^{i_3\} P_{\ell_1}^{(3)}(x) - \hat{p}'^{\{i_1} \hat{p}'^{i_2} \hat{p}'^{i_3\} P_{\ell}^{(3)}(x) \right).$$

Similar relations for tensor spherical harmonics of arbitrary rank $n+t > 3$ are contained in (18).

Equation (18) yields also information on matrix elements of operators between spinless and integer-spin states. We consider, as the simplest representative example, matrix elements of the operator L^3 . Combining (18), with $n+t=1$, with the decomposition into irreducible components of the tensor matrix elements $\langle \ell, \ell'_z | L^i L^j | \ell, \ell_z \rangle$ and $\langle \ell, \ell'_z | \hat{r}^i L^j | \ell, \ell_z \rangle$, and setting $j=3$, we get the following bilocal sums of one scalar and one vector spherical harmonic weighted by ℓ_z

$$\sum_{\ell_z=-\ell}^{\ell} \ell_z \left(Y_{\ell\ell_z}^{\ell_1 1}(\hat{p}') \right)^i Y_{\ell\ell_z}(\hat{p})^* = -\frac{1}{8\pi} \frac{2\ell+1}{\sqrt{\ell(\ell+1)}} \left(v^{\{i} v^{3\} P_{\ell}^{(2)}(x) + \hat{p}'^{\{i} \hat{p}'^{3\} P_{\ell}'(x) \right. \quad (22a)$$

$$\left. + (\hat{p}'^i \hat{p}'^3 - \hat{p}'^3 \hat{p}'^i) P_{\ell}'(x) - \frac{2}{3} \ell(\ell+1) \delta^{i3} P_{\ell}(x) \right),$$

$$\sum_{\ell_z=-\ell}^{\ell} \ell_z \left(Y_{\ell\ell_z}^{\ell_1 1}(\hat{p}') \right)^i Y_{\ell\ell_z}(\hat{p})^* = -\frac{i}{8\pi} \sqrt{2} \sqrt{\frac{2\ell+1}{\ell_1+\ell+1}} \left(\left(\hat{p}'^{\{i} v^{3\} P_{\ell_1}''(x) - \hat{p}'^{\{i} v^{3\} P_{\ell}''(x) \right) \right. \quad (22b)$$

$$\left. - ((\ell_1-\ell)(\ell+1/2)-1/2) \varepsilon^{i3p} (\hat{p}'^p P_{\ell_1}'(x) - \hat{p}^p P_{\ell}'(x)) \right).$$

Similarly, from (18) and the irreducible-tensor decompositions of $\langle \ell, \ell'_z | L^i L^j L^k | \ell, \ell_z \rangle$ and $\langle \ell, \ell'_z | \hat{r}^i L^j L^k | \ell, \ell_z \rangle$, setting $j=3=k$ we obtain bilocal sums of one scalar and one vector spherical harmonic weighted by ℓ_z^2

$$\sum_{\ell_z=-\ell}^{\ell} \ell_z^2 \left(Y_{\ell\ell_z}^{\ell_1 1}(\hat{p}') \right)^i Y_{\ell\ell_z}(\hat{p})^* = -\frac{i}{4\pi} \frac{2\ell+1}{\sqrt{\ell(\ell+1)}} \left(\frac{1}{6} \left(v^{\{i} v^{3\} v^{3\} P_{\ell}^{(3)}(x) + 3\hat{p}'^{\{i} \hat{p}'^{3\} v^{3\} P_{\ell}^{(2)}(x) \right) \right. \quad (23a)$$

$$\left. + (\hat{p}'^i \hat{p}'^3 - \hat{p}'^3 \hat{p}'^i) v^3 P_{\ell}^{(2)}(x) + \frac{1}{2} \varepsilon^{i3h} \hat{p}'^{\{h} \hat{p}'^{3\} P_{\ell}'(x) - \frac{1}{5} (\ell(\ell+1) + 1/2) v^i P_{\ell}'(x) \right.$$

$$\left. - \frac{2}{5} (\ell(\ell+1) - 3/4) v^3 \delta^{i3} P_{\ell}'(x) \right),$$

$$\sum_{\ell_z=-\ell}^{\ell} \ell_z^2 \left(Y_{\ell\ell_z}^{\ell_1 1}(\hat{p}') \right)^i Y_{\ell\ell_z}(\hat{p})^* = -\frac{1}{2\pi} \sqrt{\frac{\ell+1/2}{\ell_1+\ell+1}} \left(-\frac{1}{6} \left(\hat{p}'^{\{i} v^{3\} v^{3\} P_{\ell_1}^{(3)}(x) - \hat{p}'^{\{i} v^{3\} v^{3\} P_{\ell}^{(3)}(x) \right) \right. \quad (23b)$$

$$\left. + \hat{p}'^{\{i} \hat{p}'^{3\} \hat{p}'^{3\} P_{\ell_1}''(x) - \hat{p}'^{\{i} \hat{p}'^{3\} \hat{p}'^{3\} P_{\ell}''(x) \right) + \frac{1}{6} ((\ell_1-\ell)(\ell_1+\ell+1)-3) \varepsilon^{i3p}$$

$$\times \left(\hat{p}'^{\{i} v^{3\} P_{\ell_1}''(x) - \hat{p}'^{\{i} v^{3\} P_{\ell}''(x) \right) - \frac{1}{20} (\ell_1(\ell_1+1) - 9\ell(\ell+1) - 2) (\hat{p}'^i P_{\ell_1}'(x) - \hat{p}^i P_{\ell}'(x))$$

$$+ \frac{1}{20} (3\ell_1(\ell_1+1) - 7\ell(\ell+1) - 6) (\hat{p}'^3 P_{\ell_1}'(x) - \hat{p}^3 P_{\ell}'(x)) \delta^{i3} \right).$$

Finally, combining (18) with the irreducible-tensor decompositions of $\langle \ell, \ell'_z | L^i L^j L^k | \ell, \ell_z \rangle$, $\langle \ell, \ell'_z | \hat{r}^i L^j L^k | \ell, \ell_z \rangle$ and $\langle \ell, \ell'_z | \hat{r}^i \hat{r}^j L^k | \ell, \ell_z \rangle$, with $k=3$, we obtain bilocal sums of one scalar and one rank-2 tensor spherical

harmonic weighted by ℓ_z

$$\sum_{\ell_z=-\ell}^{\ell} \ell_z (Y_{\ell\ell_z}^{\ell_2}(\hat{p}'))^{ij} Y_{\ell\ell_z}(\hat{p})^* = -\frac{i}{2\pi} \sqrt{\frac{3}{2}} \frac{(2\ell+1)/\sqrt{\ell(\ell+1)}}{\sqrt{(2\ell-1)(2\ell+3)}} \left[\frac{1}{6} v^{\{i} v^j v^{3\}0} P_{\ell}^{(3)}(x) + \frac{1}{4} \left(-\varepsilon^{ijh} v^{\{h} v^{3\}0} \right. \right. \\ \left. \left. + 2\hat{p}'^{\{i} \hat{p}^j v^{3\}0} + \frac{4}{3} ((\hat{p}^i \hat{p}'^j - \hat{p}^j \hat{p}'^i) v^3 + (\hat{p}^j \hat{p}'^3 - \hat{p}^3 \hat{p}'^j) v^i + 2(\hat{p}^i \hat{p}'^3 - \hat{p}^3 \hat{p}'^i) v^j) \right) P_{\ell}^{(2)}(x) \right. \\ \left. + \left(\frac{1}{6} \left(-\frac{1}{2} \varepsilon^{ijh} \hat{p}'^{\{h} \hat{p}^{3\}0} + \varepsilon^{hj3} \hat{p}'^{\{h} \hat{p}^i\}0} + 2\varepsilon^{i3h} \hat{p}'^{\{h} \hat{p}^j\}0} \right) - \frac{1}{4} \varepsilon^{ijh} (\hat{p}^h \hat{p}'^3 - \hat{p}^3 \hat{p}'^h) \right. \right. \\ \left. \left. + \frac{2}{15} (\ell(\ell+1) - 3/4) \delta^{ij} v^3 - \frac{1}{5} (\ell(\ell+1) + 1/2) \delta^{j3} v^i - \frac{1}{5} (\ell(\ell+1) - 2) \delta^{i3} v^j \right) P_{\ell}'(x) \right], \quad (24a)$$

$$\sum_{\ell_z=-\ell}^{\ell} \ell_z (Y_{\ell\ell_z}^{\ell_1}(\hat{p}'))^{ij} Y_{\ell\ell_z}(\hat{p})^* = \frac{\sqrt{2\ell+1}}{\pi} \sqrt{\frac{(\ell_1+\ell-3)!!}{(\ell_1+\ell+3)!!}} \left[\frac{1}{6} (\hat{p}'^{\{i} v^j v^{3\}0} P_{\ell_1}^{(3)}(x) - \hat{p}^{\{i} v^j v^{3\}0} P_{\ell}^{(3)}(x) \right. \\ \left. + \hat{p}'^{\{i} \hat{p}^j \hat{p}^{3\}0} P_{\ell_1}^{(2)}(x) - \hat{p}'^{\{i} \hat{p}^j \hat{p}^{3\}0} P_{\ell}^{(2)}(x) \right) + \frac{1}{24} (\ell_1(\ell_1+1) - \ell(\ell+1) - 6) \left((\varepsilon^{ih3} \hat{p}'^{\{h} v^j\}0} \right. \\ \left. + \varepsilon^{jh3} \hat{p}'^{\{h} v^i\}0} P_{\ell_1}^{(2)}(x) - (\varepsilon^{ih3} \hat{p}^{\{h} v^j\}0} + \varepsilon^{jh3} \hat{p}^{\{h} v^i\}0} P_{\ell}^{(2)}(x) \right) - \frac{3}{80} (\ell_1+\ell-1)(\ell_1+\ell+3) \\ \left. \times ((\delta^{j3} \hat{p}'^i + \delta^{i3} \hat{p}'^j - 2/3 \delta^{ij} \hat{p}'^3) P_{\ell_1}'(x) - (\delta^{j3} \hat{p}^i + \delta^{i3} \hat{p}^j - 2/3 \delta^{ij} \hat{p}^3) P_{\ell}'(x)) \right], \quad (24b)$$

$$\sum_{\ell_z=-\ell}^{\ell} \ell_z (Y_{\ell\ell_z}^{\ell_2}(\hat{p}'))^{ij} Y_{\ell\ell_z}(\hat{p})^* = \frac{i}{24\pi} \sqrt{\frac{2\ell+1}{2\ell_1+1}} \frac{1}{\sqrt{\ell_1(\ell_1+1)}} \left[\hat{p}'^{\{i} \hat{p}^j v^{3\}0} P_{\ell_2}^{(3)}(x) - 2\hat{p}'^{\{i} \hat{p}^j v^{3\}0} P_{\ell_1}^{(3)}(x) \right. \\ \left. + \hat{p}^{\{i} \hat{p}^j v^{3\}0} P_{\ell}^{(3)}(x) - \frac{1}{2} ((\ell_1-\ell)(\ell_2+\ell+1) - 3) (\varepsilon^{hj3} \delta^{pi} - \varepsilon^{ih3} \delta^{pj}) \right. \\ \left. \times (\hat{p}'^{\{h} \hat{p}^p\}0} P_{\ell_2}^{(2)}(x) - 2\hat{p}'^{\{h} \hat{p}^p\}0} P_{\ell_1}^{(2)}(x) + \hat{p}^{\{h} \hat{p}^p\}0} P_{\ell}^{(2)}(x)) \right]. \quad (24c)$$

Similar results involving higher powers of ℓ_z and/or higher-rank tensor spherical harmonics follow by considering higher-rank operators. Also with the same techniques, other components of \vec{L} , instead of L^3 , can be considered, or other operators.

4.1 Local results

The previous results simplify considerably when specialized to $\hat{p}' = \hat{p}$. The local case is of interest by itself, so we discuss it in this section. The general addition theorem (18) with $\hat{p}' = \hat{p}$ reduces to

$$\sum_{\ell'_z=-\ell_n}^{\ell_n} Y_{\ell_n\ell'_z}(\hat{p}) (Y_{\ell_n\ell'_z}^{\ell(n+t)}(\hat{p}))^{i_1 \dots i_n j_1 \dots j_t *} = (-1)^n \sqrt{\frac{2\ell_n+1}{2\ell+1}} \sum_{\ell_z=-\ell}^{\ell} (Y_{\ell\ell_z}^{\ell(n+t)}(\hat{p}))^{i_1 \dots i_n j_1 \dots j_t} Y_{\ell_n\ell_z}(\hat{p})^* \\ = (-1)^{n+t} \sqrt{(2\ell+1)(2\ell_n+1)} \langle \ell, 0; n+t, 0 | \ell_n, 0 \rangle \frac{1}{\sqrt{4\pi}} (Y_{00}^{(n+t)(n+t)}(\hat{p}))^{i_1 \dots i_n j_1 \dots j_t}, \quad (25)$$

a result valid for all $\ell, n, t \geq 0$. Notice that the r.h.s. vanishes for t odd, for the same reasons as explained in connection with (17). Thus, for vector spherical harmonics, $n+t=1$ in (25), the only non-trivial result is

$$\sum_{\ell_z=-\ell}^{\ell} (Y_{\ell\ell_z}^{\ell_1}(\hat{p}))^i Y_{\ell\ell_z}(\hat{p})^* = -\frac{(\ell_1-\ell)}{4\pi} \sqrt{(\ell+1/2)(\ell_1+\ell+1)} \hat{p}^i. \quad (26)$$

For rank-2 tensor spherical harmonics, the local version of (20) is

$$\sum_{\ell_z=-\ell}^{\ell} (Y_{\ell\ell_z}^{\ell_2}(\hat{p}))^{j_1 j_2} Y_{\ell\ell_z}(\hat{p})^* = -\frac{1}{8\pi} \sqrt{\frac{3}{2}} (2\ell+1) \sqrt{\frac{\ell(\ell+1)}{(2\ell-1)(2\ell+3)}} \hat{p}^{\{j_1} \hat{p}^{j_2\}0}, \quad (27a)$$

$$\sum_{\ell_z=-\ell}^{\ell} (Y_{\ell\ell_z}^{\ell_2}(\hat{p}))^{i_1 i_2} Y_{\ell\ell_z}(\hat{p})^* = \frac{3}{16\pi} \sqrt{\ell_1(\ell_1+1)} \sqrt{\frac{2\ell+1}{\ell_2+\ell+1}} \hat{p}^{\{i_1} \hat{p}^{i_2\}0}, \quad (27b)$$

with (20b) trivially vanishing for $\hat{p}' = \hat{p}$. For rank-3 tensor spherical harmonics, the non-trivial local versions of (21) are

$$\sum_{\ell_z=-\ell}^{\ell} \left(Y_{\ell\ell_z}^{\ell_1 3}(\hat{p}) \right)^{i_1 j_1 j_2} Y_{\ell\ell_z}(\hat{p})^* = \frac{\ell_1 - \ell}{64\pi} \sqrt{\frac{10}{3}} \sqrt{2\ell+1} \sqrt{\frac{(\ell_1 + \ell - 1)(\ell_1 + \ell + 1)(\ell_1 + \ell + 3)}{(\ell_1 + \ell - 2)(\ell_1 + \ell + 4)}} \times \hat{p}^{\{i_1 \hat{p}^{j_1} \hat{p}^{j_2}\}_0}, \quad (28a)$$

$$\sum_{\ell_z=-\ell}^{\ell} \left(Y_{\ell\ell_z}^{\ell_3 3}(\hat{p}) \right)^{i_1 i_2 i_3} Y_{\ell\ell_z}(\hat{p})^* = -\frac{\ell_1 - \ell}{64\pi} \frac{5\sqrt{2}}{3} \sqrt{2\ell+1} \sqrt{\frac{(\ell_3 + \ell - 1)(\ell_3 + \ell + 1)(\ell_3 + \ell + 3)}{(\ell_3 + \ell)(\ell_3 + \ell + 2)}} \times \hat{p}^{\{i_1 \hat{p}^{i_2} \hat{p}^{i_3}\}_0}. \quad (28b)$$

The matrix elements of L^3 between one scalar and one vector spherical harmonic, with $\hat{p}' = \hat{p}$, are given by the local form of (22)

$$\sum_{\ell_z=-\ell}^{\ell} \ell_z \left(Y_{\ell\ell_z}^{\ell_1 1}(\hat{p}) \right)^i Y_{\ell\ell_z}(\hat{p})^* = -\frac{1}{8\pi} (2\ell + 1) \sqrt{\ell(\ell + 1)} (\hat{p}^i \hat{p}^3 - \delta^{i3}), \quad (29a)$$

$$\sum_{\ell_z=-\ell}^{\ell} \ell_z \left(Y_{\ell\ell_z}^{\ell_1 1}(\hat{p}) \right)^i Y_{\ell\ell_z}(\hat{p})^* = \frac{i}{8\pi} \sqrt{2} \sqrt{\frac{2\ell+1}{\ell_1 + \ell + 1}} \ell(\ell + 1) \varepsilon^{i3h} \hat{p}^h. \quad (29b)$$

For $(L^3)^2$, the local analog of (23) is

$$\sum_{\ell_z=-\ell}^{\ell} \ell_z^2 \left(Y_{\ell\ell_z}^{\ell_1 1}(\hat{p}) \right)^i Y_{\ell\ell_z}(\hat{p})^* = -\frac{i}{8\pi} (2\ell + 1) \sqrt{\ell(\ell + 1)} \varepsilon^{i3h} \hat{p}^h \hat{p}^3, \quad (30a)$$

$$\sum_{\ell_z=-\ell}^{\ell} \ell_z^2 \left(Y_{\ell\ell_z}^{\ell_1 1}(\hat{p}) \right)^i Y_{\ell\ell_z}(\hat{p})^* = \frac{(\ell_1 - \ell)}{2\pi} \sqrt{(\ell + 1/2)(\ell_1 + \ell + 1)} \left[\frac{1}{96} (\ell_1 + \ell - 1)(\ell_1 + \ell + 3) \hat{p}^{\{i \hat{p}^3 \hat{p}^3\}_0} + \frac{1}{40} (\ell_1(\ell_1 + 1) - 9\ell(\ell + 1) - 2) \hat{p}^i - \frac{1}{40} (3\ell_1(\ell_1 + 1) - 7\ell(\ell + 1) - 6) \hat{p}^3 \delta^{i3} \right]. \quad (30b)$$

Finally, for rank-2 tensor spherical harmonics, the local version of (24) yields

$$\sum_{\ell_z=-\ell}^{\ell} \ell_z \left(Y_{\ell\ell_z}^{\ell_1 2}(\hat{p}) \right)^{ij} Y_{\ell\ell_z}(\hat{p})^* = -\frac{i}{24\pi} \sqrt{\frac{3}{2}} (2\ell + 1) \sqrt{\frac{\ell(\ell + 1)}{(\ell - 1)(\ell + 3)}} \left[-\frac{1}{2} \varepsilon^{ijh} \hat{p}^{\{h \hat{p}^3\}_0} + \varepsilon^{h j 3} \hat{p}^{\{h \hat{p}^i\}_0} + 2\varepsilon^{i 3 h} \hat{p}^{\{h \hat{p}^j\}_0} \right], \quad (31a)$$

$$\sum_{\ell_z=-\ell}^{\ell} \ell_z \left(Y_{\ell\ell_z}^{\ell_1 2}(\hat{p}) \right)^{ij} Y_{\ell\ell_z}(\hat{p})^* = \frac{\ell_1 - \ell}{32\pi} \sqrt{2\ell+1} \sqrt{(\ell_1 + \ell - 1)(\ell_1 + \ell + 1)(\ell_1 + \ell + 3)} \left[\frac{1}{3} \hat{p}^{\{i \hat{p}^j \hat{p}^3\}_0} - \frac{3}{5} \left(\delta^{j3} \hat{p}^i + \delta^{i3} \hat{p}^j - \frac{2}{3} \delta^{ij} \hat{p}^3 \right) \right], \quad (31b)$$

$$\sum_{\ell_z=-\ell}^{\ell} \ell_z \left(Y_{\ell\ell_z}^{\ell_2 2}(\hat{p}) \right)^{ij} Y_{\ell\ell_z}(\hat{p})^* = -\frac{i}{32\pi} ((\ell_1 - \ell)(2\ell + 1) - 1) \sqrt{\frac{2\ell+1}{2\ell_1+1}} \sqrt{\ell_1(\ell_1 + 1)} \left(\varepsilon^{i3h} \hat{p}^{\{h \hat{p}^j\}_0} + \varepsilon^{j3h} \hat{p}^{\{h \hat{p}^i\}_0} \right). \quad (31c)$$

5 Addition theorems for spin spherical harmonics

We consider now addition theorems for spin spherical harmonics of the form

$$\sum_{j_z=-j}^j Y_{jj_z}^{\ell' s'}(\hat{r}') Y_{jj_z}^{\ell s}(\hat{r})^*, \quad (32)$$

with $s', s > 0$. Combining the factorization of orbital and spin degrees of freedom in products of two CG coefficients from sect. I.2 and the matrix elements of orbital operators from sect. I.3, we have

$$\sum_{j_z=-j}^j Y_{jj_z}^{\ell' s'}(\hat{p}') Y_{jj_z}^{\ell s}(\hat{p})^* = \sum_{\Delta=\Delta_{\min}}^{\Delta_{\max}} \frac{1}{\Delta!} C_{\ell' \ell_j}^{s' s \Delta} \langle \hat{p}' | \mathcal{P}_\ell \hat{r}^{i_1} \dots \hat{r}^{i_{|\Delta \ell|}} L^{i_{|\Delta \ell|+1}} \dots L^{i_\Delta} \rangle_0 \mathcal{P}_\ell | \hat{p} \rangle \times \langle \psi_{s'} | T^{i_1} \dots T^{i_{|\Delta \ell|}} S^{i_{|\Delta \ell|+1}} \dots S^{i_\Delta} | \psi_s \rangle. \quad (33)$$

In this equality \mathcal{P}_ℓ stands for the orbital angular momentum projector operator (see sect. I.3), $|\psi_s\rangle$ are the basis spin states entering the definition of $Y_{jj_z}^{\ell s}$ (see sect. 2), and $C_{\ell' \ell_j}^{s' s \Delta}$ and Δ_{\min}^{\max} are defined in (I.3). The orbital matrix element appearing in (33) is given explicitly for any values of their parameters in sect. I.3. For low values of $s', s > 0$, the spin matrix elements and simplified forms of the coefficients $C_{\ell' \ell_j}^{s' s \Delta}$ are given in sect. I.2. In the remainder of this section we discuss the resulting addition theorems for spin spherical harmonics. As above, throughout this section we denote $x \equiv \hat{p} \cdot \hat{p}'$ and $\vec{v} \equiv \hat{p} \wedge \hat{p}'$.

5.1 Addition theorems for spin-1/2 spherical harmonics

Spin-1/2 spherical harmonics $Y_{jj_z}^{\ell \frac{1}{2}}$, as defined by (4), vanish identically unless $j = \ell \pm 1/2$. In this subsection we always assume those values for j . In the case of two spin-1/2 harmonics with the same orbital angular momentum, (33) together with (7) with $n = 0$ yield

$$\begin{aligned} \sum_{j_z=-j}^j \left(Y_{jj_z}^{\ell \frac{1}{2}}(\hat{p}') \right)_A \left(Y_{jj_z}^{\ell \frac{1}{2}}(\hat{p}) \right)_B^* &= \frac{j+1/2}{2\ell+1} \langle \hat{p}' | \mathcal{P}_\ell | \hat{p} \rangle \delta_{AB} + \frac{2(j-\ell)}{2\ell+1} \langle \hat{p}' | L^k \mathcal{P}_\ell | \hat{p} \rangle \sigma_{AB}^k, \\ &= \frac{j+1/2}{4\pi} P_\ell(x) \delta_{AB} + i \frac{2(j-\ell)}{4\pi} P'_\ell(x) \vec{v} \cdot \vec{\sigma}_{AB}, \end{aligned} \quad (34)$$

where in the second equality we used the matrix elements (I.54) and (I.57a). Eq. (34) is the addition theorem for spin-1/2 spherical harmonics with the same orbital angular momentum. It plays an important role in the partial-wave expansion of (parity conserving) S -matrix elements between spin-1/2 states with the same intrinsic parity (see, e.g., [11]). If the intrinsic parity of the initial and final spin-1/2 states is opposite, on the other hand, a parity-conserving transition can occur only if the initial and final orbital angular momenta differ by one unit. To derive the addition theorem for spin-1/2 spherical harmonics with different orbital angular momenta we substitute (I.6) for the product of CG coefficients, (7) with $t = 0$, and (I.61a) for the resulting matrix element of \mathcal{P}_ℓ , in (33) to get

$$\sum_{j_z=-j}^j \left(Y_{jj_z}^{\ell_1 \frac{1}{2}}(\hat{p}') \right)_A \left(Y_{jj_z}^{\ell \frac{1}{2}}(\hat{p}) \right)_B^* = -\frac{\ell_1 - \ell}{4\pi} (\hat{p}' \cdot \vec{\sigma}_{AB} P'_{\ell_1}(x) - \hat{p} \cdot \vec{\sigma}_{AB} P'_\ell(x)). \quad (35)$$

This equation holds for either $\ell_1 = \ell + 1$ and $j = \ell + 1/2$, or $\ell_1 = \ell - 1$ and $j = \ell - 1/2$, otherwise the l.h.s. trivially vanishes. The local forms of these addition theorems are also of interest. Setting $\hat{p}' = \hat{p}$ in the above equalities we get,

$$\sum_{j_z=-j}^j \left(Y_{jj_z}^{\ell \frac{1}{2}}(\hat{p}) \right)_A \left(Y_{jj_z}^{\ell \frac{1}{2}}(\hat{p}) \right)_B^* = \frac{2j+1}{8\pi} \delta_{AB}, \quad \sum_{j_z=-j}^j \left(Y_{jj_z}^{\ell_1 \frac{1}{2}}(\hat{p}) \right)_A \left(Y_{jj_z}^{\ell \frac{1}{2}}(\hat{p}) \right)_B^* = -\frac{2\langle \ell \rangle + 1}{8\pi} \hat{p} \cdot \vec{\sigma}_{AB}. \quad (36)$$

Eqs. (34) and (35) exhaust the addition theorems of the form (32) for two spin-1/2 spherical harmonics.

5.2 Addition theorems for spin-1 spherical harmonics

We consider first the addition theorem for two spin-1 spherical harmonics with the same orbital angular momentum. We proceed in the same way as in the previous section, with the separation of orbital and spin quantum numbers in the product of CG coefficients carried out by means of (I.9) and the orbital matrix

elements given by (7) with $n = 0$, (I.57a) and (I.57b), to find,

$$\begin{aligned} \sum_{j_z=-j}^j (Y_{jj_z}^{\ell 1}(\hat{p}'))^r (Y_{jj_z}^{\ell 1}(\hat{p}))^{s*} &= \\ &= \frac{2\ell+1}{4\pi} \left[C_{\ell\ell j}^{110} P_\ell(x) \delta^{rs} + C_{\ell\ell j}^{111} P'_\ell(x) \hat{p}^{[r} \hat{p}^{s]} + \frac{1}{2} C_{\ell\ell j}^{112} \left(P'_\ell(x) \hat{p}^{\{r} \hat{p}^{s\}0} + P''_\ell(x) v^{\{r} v^{s\}0} \right) \right], \end{aligned} \quad (37)$$

with $C_{\ell\ell j}^{11p}$, $p = 0, 1, 2$, as given by (I.9). Eq. (37) obviously can hold only when $j = \ell$, $\ell \pm 1$, its l.h.s. trivially vanishing otherwise.

For vector spherical harmonics with orbital angular momenta differing by one unit, with the product of CG coefficients in (I.12) and the matrix elements (7) with $n = 1$, $t = 0$ and $n = 1 = t$, and (I.61a), (I.66a), we find

$$\begin{aligned} \sum_{j_z=-j}^j (Y_{jj_z}^{\ell_1 1}(\hat{p}'))^r (Y_{jj_z}^{\ell_1 1}(\hat{p}))^{s*} &= \\ &= -\frac{i}{8\pi} (\ell_1 - \ell) \left[2C_{\ell_1 \ell j}^{111} \varepsilon^{rsh} (\hat{p}^{rh} P'_{\ell_1}(x) - \hat{p}^h P'_\ell(x)) + C_{\ell_1 \ell j}^{112} (v^{\{r} \hat{p}^{s\}} P''_{\ell_1}(x) - v^{\{r} \hat{p}^{s\}} P''_\ell(x)) \right], \end{aligned} \quad (38)$$

with the coefficients $C_{\ell_1 \ell j}^{11p}$, $p = 1, 2$, defined in (I.12). We remark here that (38) holds if either $\ell_1 = \ell + 1$ and $j = \ell + 1$, ℓ , or $\ell_1 = \ell - 1$ and $j = \ell$, $\ell - 1$, otherwise the l.h.s. vanishes identically.

Similarly, for vector spherical harmonics with orbital angular momenta differing by two units, with the product of CG coefficients in eq. (I.15) and the matrix elements (7) with $n = 2$, $t = 0$ and (I.61b) we get

$$\sum_{j_z=-j}^j (Y_{jj_z}^{\ell_2 1}(\hat{p}'))^r (Y_{jj_z}^{\ell_2 1}(\hat{p}))^{s*} = -\frac{1}{4\pi} \frac{1}{\sqrt{j(j+1)}} \left(\hat{p}'^{(r} \hat{p}'^{s)0} P''_{\ell_2}(x) - \hat{p}'^{\{r} \hat{p}'^{s\}0} P''_{\ell_1}(x) + \hat{p}^{(r} \hat{p}^{s)0} P''_\ell(x) \right). \quad (39)$$

In this equation x is defined as in (37). Equation (39) holds only if either $\ell_2 = \ell + 2$ and $j = \ell_1 = \ell + 1$, or $\ell_2 = \ell - 2$ and $j = \ell_1 = \ell - 1$, the l.h.s. being identically zero in any other case.

In the local case $\hat{p}' = \hat{p}$ the above results take the form

$$\begin{aligned} \sum_{j_z=-j}^j (Y_{jj_z}^{\ell 1}(\hat{p}))^r (Y_{jj_z}^{\ell 1}(\hat{p}))^{s*} &= \frac{2\ell+1}{4\pi} \left[C_{\ell\ell j}^{110} \delta^{rs} + \frac{1}{4} \ell(\ell+1) C_{\ell\ell j}^{112} \hat{p}^{\{r} \hat{p}^{s\}0} \right], \\ \sum_{j_z=-j}^j (Y_{jj_z}^{\ell_1 1}(\hat{p}))^r (Y_{jj_z}^{\ell_1 1}(\hat{p}))^{s*} &= \frac{i}{8\pi} \sqrt{2j+1} \sqrt{2j-\langle\ell\rangle+1/2} \varepsilon^{rsh} \hat{p}^h, \\ \sum_{j_z=-j}^j (Y_{jj_z}^{\ell_2 1}(\hat{p}))^r (Y_{jj_z}^{\ell_2 1}(\hat{p}))^{s*} &= -\frac{3}{8\pi} \sqrt{j(j+1)} \hat{p}^{(r} \hat{p}^{s)0}. \end{aligned} \quad (40)$$

Eqs. (37), (38), (39) are all possible addition theorems of the form (32) for two vector spherical harmonics (see [2] for related results).

5.3 Addition theorems for spin-3/2 spherical harmonics

For spin-3/2 spherical harmonics we have four possible addition theorems, with $0 \leq \ell' - \ell \leq 3$. We begin with the case $\ell' = \ell$, for which the product of CG coefficients is given by (I.17), and the needed orbital matrix elements are (7) with $n = 0$ and (I.57a)—(I.57c). With those results we arrive at,

$$\begin{aligned} \sum_{j_z=-j}^j (Y_{jj_z}^{\ell \frac{3}{2}}(\hat{p}'))^i_A (Y_{jj_z}^{\ell \frac{3}{2}}(\hat{p}))^j_B{}^* &= \frac{2\ell+1}{4\pi} X_{AC}^{ih} \left[C_{\ell\ell j}^{\frac{3}{2} \frac{3}{2} 0} P_\ell(x) \delta^{hk} \delta_{CD} + i \frac{3}{2} C_{\ell\ell j}^{\frac{3}{2} \frac{3}{2} 1} P'_\ell(x) \delta^{hk} \vec{v} \cdot \vec{\sigma}_{CD} + \frac{3}{2} C_{\ell\ell j}^{\frac{3}{2} \frac{3}{2} 2} \right. \\ &\times \left(v^{\{h} v^{k\}0} P''_\ell(x) + \hat{p}^{\{h} \hat{p}'^{k\}0} P'_\ell(x) \right) \delta_{CD} + \frac{i}{4} C_{\ell\ell j}^{\frac{3}{2} \frac{3}{2} 3} \left(v^{\{h} v^{k\}0} P'''_\ell(x) + 3 \hat{p}^{\{h} \hat{p}'^{k\}0} P''_\ell(x) \right) \sigma_{CD}^t \left. \right] X_{DB}^{kj}, \end{aligned} \quad (41)$$

where the coefficients $C_{\ell\ell j}^{\frac{3}{2}\frac{3}{2}p}$, $p = 0, \dots, 3$ are given in (I.17b), and X_{FG}^{pq} is the projector defined in (I.B.23). Notice that when the r.h.s. of (41) is contracted with $\hat{\chi}_A^{i*}$ or $\hat{\chi}_B^j$, the corresponding projector may be dropped since it leaves the spinors invariant. We remark that (41) holds when $j = \ell \pm 3/2$, $\ell \pm 1/2$, its l.h.s. vanishing identically otherwise.

For orbital angular momenta differing by one unit, the appropriate factorization of orbital and spin degrees of freedom in CG coefficients is (I.20), and the orbital matrix elements are given by (I.62) with $t = 0$ and (I.59), (I.62) with $t = 1$ and (I.66a), and (I.62) with $t = 2$ and (I.66b). We then get,

$$\begin{aligned} \sum_{jz=-j}^j \left(Y_{jjz}^{\ell_1 \frac{3}{2}}(\hat{p}') \right)_A^i \left(Y_{jjz}^{\ell \frac{3}{2}}(\hat{p}) \right)_B^{j*} &= \frac{\ell_1 - \ell}{4\pi} X_{AC}^{ih} \left[\frac{3}{2} C_{\ell_1 \ell j}^{\frac{3}{2}\frac{3}{2}1} (\hat{p}'^q P'_{\ell_1}(x) - \hat{p}^q P'_\ell(x)) \delta^{hk} \sigma_{CD}^q \right. \\ &- i \frac{3}{2} C_{\ell_1 \ell j}^{\frac{3}{2}\frac{3}{2}2} (\hat{p}'^{\{h} v^k\}_0 P'_{\ell_1}(x) - \hat{p}^{\{h} v^k\}_0 P'_\ell(x)) \delta_{CD} + \frac{1}{4} C_{\ell_1 \ell j}^{\frac{3}{2}\frac{3}{2}3} (\hat{p}'^{\{h} v^k v^q\}_0 P'_{\ell_1}(x) - \hat{p}^{\{h} v^k v^q\}_0 P'_\ell(x)) \\ &\left. + \hat{p}'^{\{h} \hat{p}^k \hat{p}^q\}_0 P'_{\ell_1}(x) - \hat{p}^{\{h} \hat{p}^k \hat{p}^q\}_0 P'_\ell(x) \right] \sigma_{CD}^q X_{DB}^{kj}, \end{aligned} \quad (42)$$

with the coefficients $C_{\ell_1 \ell j}^{\frac{3}{2}\frac{3}{2}p}$, $p = 1, \dots, 3$ from (I.20b) and the projector X_{FG}^{pq} defined in (I.B.23). Equation (42) is valid when either $\ell_1 = \ell + 1$ and $j = \ell + 3/2, \ell + 1/2, \ell - 1/2$, or $\ell_1 = \ell - 1$ and $j = \ell + 1/2, \ell - 1/2, \ell - 3/2$, its l.h.s. being identically zero otherwise.

The case $|\ell' - \ell| = 2$ follows by using the relation among CG coefficients (I.23), (7) with $n = 2$, $t = 0$ and $t = 1$, and the matrix elements (I.61b) and (I.66b), with the result

$$\begin{aligned} \sum_{jz=-j}^j \left(Y_{jjz}^{\ell_2 \frac{3}{2}}(\hat{p}') \right)_A^i \left(Y_{jjz}^{\ell \frac{3}{2}}(\hat{p}) \right)_B^{j*} &= -\frac{1}{4\pi} \frac{1}{2\ell_1 + 1} X_{AC}^{ip} \\ &\times \left[\frac{3}{2} C_{\ell_2 \ell j}^{\frac{3}{2}\frac{3}{2}2} (\hat{p}'^{\{p} \hat{p}^r\}_0 P'_{\ell_2}(x) + \hat{p}^{\{p} \hat{p}^r\}_0 P'_\ell(x) - 2\hat{p}'^{\{p} \hat{p}^r\}_0 P'_{\ell_1}(x)) \delta_{CD} \right. \\ &\left. + \frac{i}{4} C_{\ell_2 \ell j}^{\frac{3}{2}\frac{3}{2}3} (\hat{p}'^{\{p} \hat{p}^q v^r\}_0 P'_{\ell_2}(x) + \hat{p}^{\{p} \hat{p}^q v^r\}_0 P'_\ell(x) - 2\hat{p}'^{\{p} \hat{p}^q v^r\}_0 P'_{\ell_1}(x)) \sigma_{CD}^q \right] X_{DB}^{rj}, \end{aligned} \quad (43)$$

where x , \vec{v} and X_{FG}^{pq} are as above, and $C_{\ell_2 \ell j}^{\frac{3}{2}\frac{3}{2}p}$, $p = 2, 3$, are defined in (I.23). Equation (43) is valid when either $\ell_2 = \ell + 2$ and $j = \ell + 3/2, \ell + 1/2$, or $\ell_2 = \ell - 2$ and $j = \ell - 1/2, \ell - 3/2$, its l.h.s. being identically zero otherwise.

Finally, the case $|\ell' - \ell| = 3$ follows from the CG relation (I.26) and (I.62) evaluated at $n = 3$, $t = 0$, with (I.61c). With the same notations as in the previous equations, we find

$$\begin{aligned} \sum_{jz=-j}^j \left(Y_{jjz}^{\ell_3 \frac{3}{2}}(\hat{p}') \right)_A^i \left(Y_{jjz}^{\ell \frac{3}{2}}(\hat{p}) \right)_B^{j*} &= \frac{1}{4\pi} \frac{\ell_1 - \ell}{(2\ell_1 + 1)(2\ell_2 + 1)} \sqrt{\frac{j(j+1)}{(j-1/2)(j+3/2)}} X_{AC}^{ip} \\ &\times \left[\frac{1}{3} \hat{p}'^{\{p} \hat{p}^q \hat{p}^r\}_0 P'_{\ell_3}(x) - \hat{p}'^{\{p} \hat{p}^q \hat{p}^r\}_0 P'_{\ell_2}(x) + \hat{p}'^{\{p} \hat{p}^q \hat{p}^r\}_0 P'_{\ell_1}(x) - \frac{1}{3} \hat{p}^{\{p} \hat{p}^q \hat{p}^r\}_0 P'_\ell(x) \right] \sigma_{CD}^q X_{DB}^{rj}, \end{aligned} \quad (44)$$

which is valid for either $\ell_3 = \ell + 3$ and $j = \ell + 3/2$, or $\ell_3 = \ell - 3$ and $j = \ell - 3/2$. For other values of j , the l.h.s. vanishes.

We quote here also the local forms of the previous results. With $\hat{p}' = \hat{p}$ we have

$$\begin{aligned}
\sum_{j_z=-j}^j \left(Y_{jj_z}^{\ell \frac{3}{2}}(\hat{p}) \right)_A^i \left(Y_{jj_z}^{\ell \frac{3}{2}}(\hat{p}) \right)_B^{j*} &= \frac{2\ell+1}{4\pi} X_{AC}^{ih} \left[C_{\ell\ell j}^{\frac{3}{2}\frac{3}{2}0} \delta^{hk} + \frac{3}{4} \ell(\ell+1) C_{\ell\ell j}^{\frac{3}{2}\frac{3}{2}2} \hat{p}^{\{h\hat{p}^k\}_0} \right] X_{CB}^{kj}, \\
\sum_{j_z=-j}^j \left(Y_{jj_z}^{\ell_1 \frac{3}{2}}(\hat{p}) \right)_A^i \left(Y_{jj_z}^{\ell \frac{3}{2}}(\hat{p}) \right)_B^{j*} &= \frac{1}{16\pi} (\ell_1 + \ell + 1) X_{AC}^{ih} \left[3 C_{\ell_1 \ell j}^{\frac{3}{2}\frac{3}{2}1} \hat{p}^q \delta^{hk} \right. \\
&\quad \left. + \frac{1}{16} C_{\ell_1 \ell j}^{\frac{3}{2}\frac{3}{2}3} (\ell_1 + \ell - 1)(\ell_1 + \ell + 3) \hat{p}^{\{h\hat{p}^k\hat{p}^q\}_0} \right] \sigma_{CD}^q X_{DB}^{kj}, \\
\sum_{j_z=-j}^j \left(Y_{jj_z}^{\ell_2 \frac{3}{2}}(\hat{p}) \right)_A^i \left(Y_{jj_z}^{\ell \frac{3}{2}}(\hat{p}) \right)_B^{j*} &= -\frac{9}{16\pi} \frac{\ell_1(\ell_1+1)}{2\ell_1+1} C_{\ell_2 \ell j}^{\frac{3}{2}\frac{3}{2}2} X_{AC}^{ip} \hat{p}^{\{p\hat{p}^r\}_0} X_{CB}^{rj}, \\
\sum_{j_z=-j}^j \left(Y_{jj_z}^{\ell_3 \frac{3}{2}}(\hat{p}) \right)_A^i \left(Y_{jj_z}^{\ell \frac{3}{2}}(\hat{p}) \right)_B^{j*} &= \frac{5}{192\pi} \frac{\sqrt{2j+1}}{(2\ell_1+1)(2\ell_2+1)} \sqrt{\frac{(2j+3)!}{(2j-2)!}} X_{AC}^{ip} \hat{p}^{\{p\hat{p}^q\hat{p}^r\}_0} \sigma_{CD}^q X_{DB}^{rj}.
\end{aligned} \tag{45}$$

With (41)–(44) established, there are no further non-trivial relations of the form (32) for two spin-3/2 spherical harmonics.

5.4 Addition theorems for one spin-0 and one spin-1 spherical harmonics

Using the factorization of orbital and spin degrees of freedom in products of CG coefficients provided by (I.32)–(I.35), we recover the results already given in (19).

5.5 Addition theorems for one spin-1/2 and one spin-3/2 spherical harmonics

For $s = 1/2$ and $s' = 3/2$, with $\ell' = \ell$, the relation among CG coefficients (I.37) or (I.40), and (7) with $n = 0$ and $t = 1, 2$, together with (I.57a) and (I.57b), lead to,

$$\begin{aligned}
\sum_{j_z=-j}^j \left(Y_{jj_z}^{\ell \frac{3}{2}}(\hat{p}') \right)_A^i \left(Y_{jj_z}^{\ell \frac{1}{2}}(\hat{p}) \right)_B^{j*} &= \frac{2\ell+1}{8\pi} \sqrt{\frac{3}{2}} X_{AC}^{ik} \left[C_{\ell\ell j}^{\frac{3}{2}\frac{1}{2}1} i v^k P'_\ell(x) \delta_{CB} \right. \\
&\quad \left. - \frac{1}{4} C_{\ell\ell j}^{\frac{3}{2}\frac{1}{2}2} \left(v^{\{k v^j\}_0} P''_\ell(x) + \hat{p}^{\{k\hat{p}^j\}_0} P'_\ell(x) \right) \sigma_{CB}^j \right],
\end{aligned} \tag{46}$$

with $C_{\ell_2 \ell j}^{\frac{3}{2}\frac{1}{2}1,2}$ given by (I.37b) and X_{FG}^{pq} by (I.B.23). Notice that the subindex “0” in the second term on the r.h.s. of (46) is actually not needed, because $X_{AC}^{ik} \sigma_{CB}^k \equiv 0$. The case with $s = 3/2$ and $s' = 1/2$ follows immediately from (46) by complex conjugation and the exchange $\hat{p} \leftrightarrow \hat{p}'$. The equality in (46) holds for $j = \ell \pm 1/2$, otherwise $Y_{jj_z}^{\ell \frac{1}{2}}(\hat{p}) \equiv 0$.

The case $|\ell' - \ell| = 1$ follows from (I.42), (7) with $n = 1, t = 0$ and (I.61a), and (7) with $n = 1, t = 1$ and (I.66a)

$$\begin{aligned}
\sum_{j_z=-j}^j \left(Y_{jj_z}^{\ell_1 \frac{3}{2}}(\hat{p}') \right)_A^i \left(Y_{jj_z}^{\ell \frac{1}{2}}(\hat{p}) \right)_B^{j*} &= \frac{\ell_1 - \ell}{8\pi} \sqrt{\frac{3}{2}} X_{AC}^{ik} \left[C_{\ell_1 \ell j}^{\frac{3}{2}\frac{1}{2}1} (\hat{p}'^k P'_{\ell_1}(x) - \hat{p}^k P'_\ell(x)) \delta_{CB} \right. \\
&\quad \left. + \frac{i}{4} C_{\ell_1 \ell j}^{\frac{3}{2}\frac{1}{2}2} \left(\hat{p}'^{\{k v^p\}} P''_{\ell_1}(x) - \hat{p}^{\{k v^p\}} P''_\ell(x) \right) \sigma_{CB}^p \right],
\end{aligned} \tag{47}$$

with the coefficients $C_{\ell_1 \ell j}^{\frac{3}{2}\frac{1}{2}1,2}$ given by (I.42b), with $\Delta s = 1$. Eq. (47) holds for $j = \ell \pm 1/2$, its l.h.s. vanishing otherwise. Similarly,

$$\begin{aligned}
\sum_{j_z=-j}^j \left(Y_{jj_z}^{\ell_1 \frac{1}{2}}(\hat{p}') \right)_A^i \left(Y_{jj_z}^{\ell \frac{3}{2}}(\hat{p}) \right)_B^{j*} &= \frac{\ell_1 - \ell}{8\pi} \sqrt{\frac{3}{2}} \left[C_{\ell_1 \ell j}^{\frac{1}{2}\frac{3}{2}1} (\hat{p}'^k P'_{\ell_1}(x) - \hat{p}^k P'_\ell(x)) \delta_{AC} \right. \\
&\quad \left. + \frac{i}{4} C_{\ell_1 \ell j}^{\frac{1}{2}\frac{3}{2}2} \left(\hat{p}'^{\{k v^p\}} P''_{\ell_1}(x) - \hat{p}^{\{k v^p\}} P''_\ell(x) \right) \sigma_{AC}^p \right] X_{CB}^{ki},
\end{aligned} \tag{48}$$

where now $C_{\ell_1 \ell_j}^{\frac{1}{2} \frac{3}{2} 1, 2}$ are given by (I.42b) with $\Delta s = -1$, and the equality holds for $j = \ell_1 \pm 1/2$.

Finally, the last case to be considered is $\ell' - \ell = \pm 2$. In this case we use the factorization of orbital and spin degrees of freedom from (I.47), together with (7) with $n = 2$, $t = 1$ and (I.61b). In this way we get,

$$\sum_{j_z = -j}^j \left(Y_{jjz}^{\ell_2 \frac{3}{2}}(\hat{p}') \right)_A^i \left(Y_{jjz}^{\ell \frac{1}{2}}(\hat{p}) \right)_B^{i*} = \frac{(\ell_1 - \ell)}{4\pi} \sqrt{\frac{j+1/2}{\ell_1(\ell_1+1)(2\ell_1+1)}} X_{AC}^{ik} \times \left[\hat{p}'^{(k} \hat{p}^{p)0} P_{\ell_2}''(x) - \hat{p}'^{\{k} \hat{p}^{p\}0} P_{\ell_1}''(x) + \hat{p}^{(k} \hat{p}^{p)0} P_{\ell}''(x) \right] \sigma_{CB}^p, \quad (49)$$

with X_{AC}^{ik} defined as in the previous equations. The result (49) applies if either $\ell_2 = \ell + 2$ and $j = \ell + 1/2$, or $\ell_2 = \ell - 2$ and $j = \ell - 1/2$, otherwise its l.h.s. trivially vanishes. Similarly, we can derive,

$$\sum_{j_z = -j}^j \left(Y_{jjz}^{\ell_2 \frac{1}{2}}(\hat{p}') \right)_A^i \left(Y_{jjz}^{\ell \frac{3}{2}}(\hat{p}) \right)_B^{i*} = -\frac{(\ell_1 - \ell)}{4\pi} \sqrt{\frac{j+1/2}{\ell_1(\ell_1+1)(2\ell_1+1)}} \sigma_{AC}^k \times \left[\hat{p}'^{(k} \hat{p}^{p)0} P_{\ell_2}''(x) - \hat{p}'^{\{k} \hat{p}^{p\}0} P_{\ell_1}''(x) + \hat{p}^{(k} \hat{p}^{p)0} P_{\ell}''(x) \right] X_{CB}^{pi}, \quad (50)$$

the extra minus sign in the r.h.s., with respect to (49), coming from the factor Δs in (I.47b). Now (50) holds if either $\ell_2 = \ell + 2$ and $j = \ell + 3/2$, or $\ell_2 = \ell - 2$ and $j = \ell - 3/2$.

Setting $\hat{p}' = \hat{p}$ in the above equalities, we obtain the local sums

$$\begin{aligned} \sum_{j_z = -j}^j \left(Y_{jjz}^{\ell_2 \frac{3}{2}}(\hat{p}') \right)_A^i \left(Y_{jjz}^{\ell \frac{1}{2}}(\hat{p}) \right)_B^{i*} &= -\sqrt{\frac{3}{2}} \frac{2\ell+1}{64\pi} \ell(\ell+1) C_{\ell \ell_j}^{\frac{3}{2} \frac{1}{2} 2} X_{AC}^{ik} \hat{p}^{\{k} \hat{p}^{j\}0} \sigma_{CB}^j, \\ \sum_{j_z = -j}^j \left(Y_{jjz}^{\ell_1 \frac{3}{2}}(\hat{p}') \right)_A^i \left(Y_{jjz}^{\ell \frac{1}{2}}(\hat{p}) \right)_B^{i*} &= \frac{1}{16\pi} \sqrt{\frac{3}{2}} (\ell_1 + \ell + 1) C_{\ell_1 \ell_j}^{\frac{3}{2} \frac{1}{2} 1} X_{AB}^{ik} \hat{p}^k, \\ \sum_{j_z = -j}^j \left(Y_{jjz}^{\ell_1 \frac{1}{2}}(\hat{p}') \right)_A^i \left(Y_{jjz}^{\ell \frac{3}{2}}(\hat{p}) \right)_B^{i*} &= \frac{1}{16\pi} \sqrt{\frac{3}{2}} (\ell_1 + \ell + 1) C_{\ell_1 \ell_j}^{\frac{1}{2} \frac{3}{2} 1} \hat{p}^k X_{AB}^{ki}, \\ \sum_{j_z = -j}^j \left(Y_{jjz}^{\ell_2 \frac{3}{2}}(\hat{p}') \right)_A^i \left(Y_{jjz}^{\ell \frac{1}{2}}(\hat{p}) \right)_B^{i*} &= \frac{3}{16\pi} (\ell_1 - \ell) \sqrt{\frac{(j+1/2)\ell_1(\ell_1+1)}{2\ell_1+1}} X_{AC}^{ik} \hat{p}^{\{k} \hat{p}^{p\}0} \sigma_{CB}^p, \\ \sum_{j_z = -j}^j \left(Y_{jjz}^{\ell_2 \frac{1}{2}}(\hat{p}') \right)_A^i \left(Y_{jjz}^{\ell \frac{3}{2}}(\hat{p}) \right)_B^{i*} &= -\frac{3}{16\pi} (\ell_1 - \ell) \sqrt{\frac{(j+1/2)\ell_1(\ell_1+1)}{2\ell_1+1}} \sigma_{AC}^p \hat{p}^{\{p} \hat{p}^k\}0} X_{CB}^{ki}. \end{aligned} \quad (51)$$

No further non-trivial addition theorems of the form (32) can be given for one spin-1/2 and one spin-3/2 spherical harmonics.

6 Final Remarks

Using the results of I we developed here a systematic approach to the derivation of addition theorems for spin spherical harmonics. The results of our approach are appropriately summarized by the general expressions (7), (8) for bilocal sums of spherical harmonics, which are also explicit expressions for bilocal spherical harmonics, the general form (18) of the addition theorem for one scalar and one tensor spherical harmonics, and the general form (33) of the addition theorem for two spin spherical harmonics. Those results are based on the irreducible-tensor matrix elements from sect. I.3. Bilocal sums of a more general type can be obtained by means of reducible matrix elements, as illustrated by (10)–(16) and (22)–(24).

In sect. 3 we give results for bilocal sums of ordinary spherical harmonics with certain weight functions depending on ℓ_z . Many other classes of sums can in principle be obtained with the same techniques, by considering other matrix elements. In sect. 4 we explicitly state addition theorems for and other bilocal sums of one scalar and one tensor spherical harmonics, up to rank 3. Results for higher ranks can equally well be obtained, although the involved amount of algebraic labor is a rapidly increasing function of rank. The local versions of those results are also of interest, and are summarized in sect. 4.1. Addition theorems

for spin spherical harmonics of the form (32) are given in sect. 5, together with their local forms, for spins $1/2 \leq s' = s \leq 3/2$, and $(s', s) = (3/2, 1/2)$ and $(1/2, 3/2)$. Similar results for arbitrary values of s', s can be obtained as particular cases of (33) with the coefficients given by (I.3) and the matrix elements from sect. I.3 and app. I.C. As in the previous case, however, for higher spins ($s', s > 2$) the algebraic complexity of the results quickly grows unmanageable. Clearly, for spins larger than 2 an automated procedure would be needed. Even though there is no pressing physical need for such further generalizations, because higher spin states occur infrequently in nature, they could lead to improved calculational tools that would be of interest even in phenomenological contexts.

The addition theorems for spin spherical harmonics given in sections 4 and 5 are directly applicable to the computation of partial-wave expansions of S -matrix elements for two-body scattering of particles with spin. As discussed in the introduction that is our main motivation for this paper, but we hope our results to be more widely applicable.

References

- [1] L. C. Biendenharn, J. D. Louck, “Angular Momentum in Quantum Physics,” Encyclopedia of Mathematics and its Applications Vol. 8, Cambridge Univ. Press, New York, 1985.
- [2] D. A. Varshalovich, A. N. Moskalev, V. K. Khersonskii, “Quantum Theory of Angular Momentum,” World Scientific, Singapore, 1988.
- [3] J. D. Jackson, “Classical Electrodynamics,” John Wiley, New York, 1999.
- [4] A. O. Bouzas, Eur. Phys. J. A **43** (2010) 351-368.
- [5] A. O. Bouzas, “Addition theorems for spin spherical harmonics. I Preliminaries,” to be published.
- [6] A. Galindo, P. Pascual, “Quantum Mechanics,” Vol. I, Springer-Verlag, New York, 1990.
- [7] A. R. Edmonds, “Angular Momentum in Quantum Mechanics,” Princeton Univ. Press, Princeton, 1996.
- [8] E. P. Wigner, “Group Theory and its Application to the Quantum Mechanics of Atomic Spectra,” Academic Press, New York, 1959.
- [9] B. R. Judd, “Angular Momentum Theory for Diatomic Molecules,” Academic Press, New York, 1975.
- [10] L. C. Biendenharn, J. D. Louck, “The Racah-Wigner algebra in Quantum Theory,” Encyclopedia of Mathematics and its Applications Vol. 9, Cambridge Univ. Press, New York, 1985.
- [11] O. Nachtmann, “Elementary Particle Physics,” Springer-Verlag, New York, 1990.

A Local sums of two spherical harmonics

In this appendix we list several particular cases of (17), which are local versions of the bilocal sums of sect. 3. Some of these relations are familiar from textbooks and most, if not all, of them are certainly well known. We put them together here for reference (see also [2]).

For $n = 0 = t$, (17) reduces to the familiar equality $\sum_{m=-\ell}^{\ell} |Y_{\ell m}(\theta, \varphi)|^2 = (2\ell + 1)/(4\pi)$. By setting $n = 0 = m$ in (17) with $t > 0$ even, and proceeding as discussed in relation to (10) we obtain

$$\sum_{\ell_z=-\ell}^{\ell} |Y_{\ell \ell_z}(\hat{r})|^2 \ell_z^2 = \frac{1}{8\pi} (2\ell + 1) \ell (\ell + 1) (\sin \theta)^2, \quad (\text{A.1a})$$

$$\sum_{\ell_z=-\ell}^{\ell} |Y_{\ell \ell_z}(\hat{r})|^2 \ell_z^4 = \frac{1}{8\pi} (2\ell + 1) \ell (\ell + 1) (\sin \theta)^2 \left(\frac{3}{4} (\ell (\ell + 1) - 2) (\sin \theta)^2 + 1 \right), \quad (\text{A.1b})$$

$$\sum_{\ell_z=-\ell}^{\ell} |Y_{\ell \ell_z}(\hat{r})|^2 \ell_z^6 = \frac{1}{2^9 \pi} (2\ell + 1) \ell (\ell + 1) (\sin \theta)^2 \left(\frac{5}{\ell (\ell + 1)} \frac{(\ell + 3)!}{(\ell - 3)!} \cos(4\theta) \right)$$

$$-20 \frac{(\ell+2)!}{(\ell-2)!} \cos(2\theta) + 15\ell^2(\ell+1)^2 + 4 \Big), \quad (\text{A.1c})$$

$$\begin{aligned} \sum_{\ell_z=-\ell}^{\ell} |Y_{\ell\ell_z}(\hat{r})|^2 \ell_z^8 &= \frac{35}{2^{14}\pi} (2\ell+1)(\sin\theta)^2 \left(-\frac{(\ell+4)!}{(\ell-4)!} \cos(6\theta) + 2\frac{(\ell+3)!}{(\ell-3)!} (3\ell(\ell+1)-4) \cos(4\theta) \right. \\ &\quad - \frac{1}{5} \frac{(\ell+2)!}{(\ell-2)!} (75\ell^2(\ell+1)^2 - 70\ell(\ell+1) + 24) \cos(2\theta) \\ &\quad \left. + \frac{2}{35} \ell(\ell+1)(175\ell^3(\ell+1)^3 - 140\ell^2(\ell+1)^2 + 84\ell(\ell+1) + 16) \right). \end{aligned} \quad (\text{A.1d})$$

Higher moments of ℓ_z ($t > 8$) can be computed, but the labor involved increases rapidly with t . The equalities (A.1a) and (A.1b) are just the particularization to $\hat{r}' = \hat{r}$ of (10b) and (10d), resp. When $m = \pm t$, with $t > 0$ even, (17) acquires the explicit form

$$\begin{aligned} \sum_{\ell_z=-\ell}^{\ell} Y_{\ell(\ell_z \pm t)}(\hat{r}) Y_{\ell\ell_z}(\hat{r})^* \sqrt{\frac{(\ell \mp \ell_z)!}{(\ell \mp \ell_z - t)!}} \sqrt{\frac{(\ell \pm \ell_z + t)!}{(\ell \pm \ell_z)!}} &= \\ &= \frac{(-1)^{t/2}}{4\pi} \frac{(t-1)!!}{2^{t/2}(t/2)!} (2\ell+1) \frac{(\ell+t/2)!}{(\ell-t/2)!} (\sin\theta)^t e^{\pm it\varphi}. \end{aligned} \quad (\text{A.2})$$

For odd t the l.h.s. of this equation vanishes, in agreement with (17).

For $t = 0$, $n = 1$ (17) reduces to,

$$\sum_{\ell_z=-\ell}^{\ell} Y_{(\ell+1)\ell_z}(\hat{p}) Y_{\ell\ell_z}(\hat{p})^* \sqrt{(\ell+1)^2 - \ell_z^2} = \frac{1}{4\pi} \sqrt{2\ell+1} \sqrt{2\ell+3} (\ell+1) \cos\theta, \quad (\text{A.3a})$$

$$\sum_{\ell_z=-\ell}^{\ell} Y_{(\ell+1)(\ell_z+m)}(\hat{p}) Y_{\ell\ell_z}(\hat{p})^* \sqrt{\ell+1+m\ell_z} \sqrt{\ell+2+m\ell_z} = -\frac{m}{4\pi} \sqrt{2\ell+1} \sqrt{2\ell+3} (\ell+1) \sin\theta e^{im\varphi}, \quad (\text{A.3b})$$

where on the second line $m = \pm 1$. We restricted ourselves to $\ell' = \ell+1$ in (A.3), since the case $\ell' = \ell-1$ can be obtained easily from there by appropriate transformation of parameters. Similarly, for $t = 0$, $n = 2$ we have

$$\begin{aligned} \sum_{\ell_z=-\ell+1}^{\ell-1} Y_{(\ell+1)\ell_z}(\hat{p}) Y_{(\ell-1)\ell_z}(\hat{p})^* \sqrt{(\ell^2 - \ell_z^2)((\ell+1)^2 - \ell_z^2)} &= \\ &= \frac{1}{8\pi} \sqrt{(2\ell-1)(2\ell+3)} \ell(\ell+1) (3(\cos\theta)^2 - 1), \end{aligned} \quad (\text{A.4a})$$

$$\begin{aligned} \sum_{\ell_z=-\ell+1}^{\ell-1} Y_{(\ell+1)(\ell_z+m)}(\hat{p}) Y_{(\ell-1)\ell_z}(\hat{p})^* \sqrt{(\ell^2 - \ell_z^2)} \sqrt{(\ell+m\ell_z+1)(\ell+m\ell_z+2)} &= \\ &= -\frac{3}{8\pi} \sqrt{(2\ell-1)(2\ell+3)} \ell(\ell+1) m \cos\theta \sin\theta e^{im\varphi}, \end{aligned} \quad (\text{A.4b})$$

$$\begin{aligned} \sum_{\ell_z=-\ell+1}^{\ell-1} Y_{(\ell+1)(\ell_z+2m)}(\hat{p}) Y_{(\ell-1)\ell_z}(\hat{p})^* \sqrt{\frac{(\ell+m\ell_z+3)!}{(\ell+m\ell_z-1)!}} &= \\ &= \frac{3}{8\pi} \sqrt{(2\ell-1)(2\ell+3)} \ell(\ell+1) (\sin\theta)^2 e^{2im\varphi}, \end{aligned} \quad (\text{A.4c})$$

with $m = \pm 1$ both in (A.4b) and (A.4c). Equation (A.4a) is well known [2], equations (A.4b), (A.4c) extend it to $\ell'_z = \ell_z \pm 1$ and $\ell_z \pm 2$, resp., and (13) to $\hat{p}' \neq \hat{p}$.

Setting $n = 1 = t$ in (17) we obtain

$$\begin{aligned} \sum_{\ell_z=-\ell}^{\ell} Y_{(\ell+1)(\ell_z+m)}(\hat{p}) Y_{\ell\ell_z}(\hat{p})^* \ell_z \sqrt{(\ell+1+m\ell_z)(\ell+2+m\ell_z)} &= \\ &= -\frac{1}{8\pi} \sqrt{2\ell+1} \sqrt{2\ell+3} \ell(\ell+1) \sin\theta e^{im\varphi}, \end{aligned} \quad (\text{A.5})$$

with $m = \pm 1$, whereas the case $m = 0$ leads to a trivial identity. For $n = 1$, $t = 2$ with $m = 0, \pm 1$, from (17) we have

$$\begin{aligned} \sum_{\ell_z = -\ell}^{\ell} Y_{(\ell+1)(\ell_z+m)}(\hat{p}) Y_{\ell\ell_z}(\hat{p})^* \ell_z^2 \sqrt{(\ell+1+m\ell_z)(\ell+2+m\ell_z)} = \\ = \frac{m}{32\pi} \sqrt{(2\ell+1)(2\ell+3)} \ell(\ell+1) ((\ell+2)\sin(3\theta) - (3\ell+2)\sin\theta) e^{im\varphi}, \end{aligned} \quad (\text{A.6a})$$

$$\sum_{\ell_z = -\ell}^{\ell} Y_{(\ell+1)\ell_z}(\hat{p}) Y_{\ell\ell_z}(\hat{p})^* \ell_z^2 \sqrt{(\ell+1)^2 - \ell_z^2} = \frac{1}{8\pi} \sqrt{(2\ell+1)(2\ell+3)} \frac{(\ell+2)!}{(\ell-1)!} \cos\theta (\sin\theta)^2. \quad (\text{A.6b})$$

In (A.5), (A.6) we specialized the general form (17) to $\ell_1 = \ell + 1$, the case $\ell_1 = \ell - 1$ following from them by redefinition of parameters. Those results are to be compared with (A.3). Similarly, for $n = 2$, $t = 1$ from (17) we get

$$\begin{aligned} \sum_{\ell_z = -\ell+1}^{\ell-1} Y_{(\ell+1)(\ell_z+m)}(\hat{p}) Y_{(\ell-1)\ell_z}(\hat{p})^* \ell_z \sqrt{\ell^2 - \ell_z^2} \sqrt{(\ell+1+m\ell_z)(\ell+2+m\ell_z)} = \\ = -\frac{1}{8\pi} \sqrt{(2\ell-1)(2\ell+3)} \frac{(\ell+1)!}{(\ell-1)!} \cos\theta \sin\theta e^{im\varphi}, \end{aligned} \quad (\text{A.7a})$$

$$\begin{aligned} \sum_{\ell_z = -\ell+1}^{\ell-1} Y_{(\ell+1)(\ell_z+2m)}(\hat{p}) Y_{(\ell-1)\ell_z}(\hat{p})^* \ell_z \sqrt{\frac{(\ell+m\ell_z+3)!}{(\ell+m\ell_z-1)!}} = \\ = \frac{m}{4\pi} \sqrt{(2\ell-1)(2\ell+3)} \frac{(\ell+1)!}{(\ell-1)!} (\sin\theta)^2 e^{2im\varphi}, \end{aligned} \quad (\text{A.7b})$$

with $m = \pm 1$. Finally, for $n = 2 = t$ (17) yields

$$\begin{aligned} \sum_{\ell_z = -\ell+1}^{\ell-1} Y_{(\ell+1)\ell_z}(\hat{p}) Y_{(\ell-1)\ell_z}(\hat{p})^* \ell_z^2 \sqrt{\ell^2 - \ell_z^2} \sqrt{((\ell+1)^2 - \ell_z^2)} = \\ = \frac{1}{64\pi} \sqrt{(2\ell-1)(2\ell+3)} \frac{(\ell+2)!}{(\ell-2)!} (\sin\theta)^2 (5\cos(2\theta) + 3), \end{aligned} \quad (\text{A.8a})$$

$$\begin{aligned} \sum_{\ell_z = -\ell+1}^{\ell-1} Y_{(\ell+1)(\ell_z+m)}(\hat{p}) Y_{(\ell-1)\ell_z}(\hat{p})^* \ell_z^2 \sqrt{\ell^2 - \ell_z^2} \sqrt{(\ell+1+m\ell_z)(\ell+2+m\ell_z)} = \\ = \frac{m}{32\pi} \sqrt{(2\ell-1)(2\ell+3)} \frac{(\ell+1)!}{(\ell-1)!} \cos\theta \sin\theta (5(\ell+2)(\cos\theta)^2 - 5\ell - 6) e^{im\varphi}, \end{aligned} \quad (\text{A.8b})$$

$$\begin{aligned} \sum_{\ell_z = -\ell+1}^{\ell-1} Y_{(\ell+1)(\ell_z+2m)}(\hat{p}) Y_{(\ell-1)\ell_z}(\hat{p})^* \ell_z^2 \sqrt{\frac{(\ell+3+m\ell_z)!}{(\ell-1+m\ell_z)!}} = \\ = -\frac{1}{32\pi} \sqrt{(2\ell-1)(2\ell+3)} \frac{(\ell+1)!}{(\ell-1)!} (\sin\theta)^2 (5(\ell+2)(\cos\theta)^2 - 7\ell + 2) e^{2im\varphi}, \end{aligned} \quad (\text{A.8c})$$

where in the last two equalities $m = \pm 1$. Many other particular addition theorems can be derived from the general form (17). We will refrain from further discussing them here.